Tour of the Gyrokinetic Equation

Part 2

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In $\boxed{\mathcal{E}} = \frac{mv^2}{2} + e\phi$, $\boxed{\mu} = \frac{mv_\perp^2}{2B}$, $\boxed{\vec{R}} = \vec{r} + \underbrace{\frac{\hat{b}\times\vec{v}}{\Omega}}_{\vec{\rho}}$ coordinates

and distribution function $\boxed{g} = f - f_0$, $f_0 = \frac{N e^{-\frac{mv^2}{2T}}}{(2\pi T/m)^{3/2}} \cdot \left(1 - \frac{e\phi}{T}\right)$

$\boxed{\text{Gyrokinetic equation (nonlinear)}}$: $\boxed{\frac{\partial g}{\partial t} + (v_\parallel \hat{b} + \vec{v}_E + \vec{v}_d)\cdot\nabla g = \vec{v}_E \cdot \nabla f_0 + \frac{e}{T}\left\langle\frac{\partial \phi}{\partial t}\right\rangle f_0}$

$$\vec{v}_E = \frac{\hat{b}\times\nabla\langle\phi\rangle}{B}, \quad \vec{v}_d = \frac{\vec{b}}{\Omega}\times\left[\frac{v_\perp^2}{2}\nabla \ln B + v_\parallel^2(\vec{b}\cdot\nabla)\vec{b}\right]$$

→ gyroaverage $\langle\phi\rangle = \int_0^{2\pi} \phi(\vec{r})\frac{d\theta}{2\pi}$ done at constant $\vec{R}, \mathcal{E}, \mu$

- Fourier Transform $\phi(\vec{r}) = \int d\vec{k}\, \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{e^{i\vec{k}\cdot\vec{R}}\, e^{-i\vec{k}\cdot\vec{\rho}}} \phi(\vec{k})$, $\quad e^{-i\vec{k}\cdot\vec{\rho}} \longrightarrow e^{-ik_\perp\rho\cos(\theta)}$

$$\Rightarrow \langle\phi\rangle = \int d\vec{k}\, e^{i\vec{k}\cdot\vec{R}}\phi(\vec{k}) \boxed{\int_0^{2\pi}\frac{d\theta}{2\pi} e^{-ik_\perp\rho\cos(\theta)}} = J_0(k_\perp\rho)$$

Therefore, in $\vec{k}$ space, $\langle\phi\rangle \to J_0\phi$

$\nabla_\parallel = \vec{b}\cdot\nabla$

$\boxed{\text{Gyrokinetic equation (Fourier-transformed, Linearized)}}$: $\boxed{v_\parallel \nabla_\parallel g_a - i(\omega - \omega_d) g_a = -\frac{ie}{T} J_0 \phi (\omega - \omega_*^T) f_0}$

$$\omega_d = \vec{k}_\perp \cdot \vec{v}_d, \quad \omega_*^T = \omega_*\left[1 + \eta\left(\frac{mv^2}{2T} - \frac{3}{2}\right)\right], \quad \eta = \frac{d_q \ln T}{d_q \ln n}$$

$\hookrightarrow T\frac{k_\alpha}{e} d_q \ln n$

What if we included collisions?

add term to GK eq. $+\langle C(f)\rangle$

example: Lenard-Bernstein operator $C(f) = \nu \frac{\partial}{\partial \vec{v}}\cdot\left(\vec{v}f + \frac{\partial f}{\partial \vec{v}}\right)$

need to gyroaverage! In Fourier space,

$$\rightarrow C(f) = \nu \int d\vec{k}\, \frac{\partial}{\partial \vec{v}}\cdot\left(\vec{v}\, f(\vec{k},\vec{v})e^{i\vec{k}\cdot\vec{\rho}} + \frac{\partial}{\partial \vec{v}}\left[f(\vec{k},\vec{v})e^{i\vec{k}\cdot\vec{\rho}}\right]\right)e^{i\vec{k}\cdot\vec{R}}$$

- as $\vec{v}$ depends on θ, ρ depends on $\vec{v}$, we need to approximate
  $f = f_0 + g \Rightarrow C(f) \simeq C(g)$
- coordinate transformation $\vec{v} \rightarrow \mu, v_\parallel, \theta$ leads to

$$\boxed{\langle C(g)\rangle = \nu\left[\frac{\partial}{\partial v_\parallel}\left(\frac{\partial}{\partial v_\parallel} + v_\parallel\right) + 2\frac{\partial}{\partial \mu}\left(\frac{\mu}{B}\frac{\partial}{\partial \mu} + \mu\right) + k_\perp^2 \rho_s^2\right]g}, \quad \rho_s = \frac{v_{th}}{\sqrt{2}} = \frac{\sqrt{T/m}}{eB/m}$$

What if we include electromagnetic perturbations?

$$\phi \longrightarrow \chi = \phi - \vec{v}\cdot\vec{A}$$

↳ electromagnetic vector potential
$\vec{B} = \nabla \times \vec{A}$

as $\vec{v} = v_\parallel \vec{b} + \vec{v}_\perp$, and ϕ only appears as $\langle\phi\rangle$

$$\langle\phi\rangle \longrightarrow \langle\chi\rangle = \langle\phi\rangle - v_\parallel\langle A_\parallel\rangle - \langle\vec{v}_\perp\cdot\vec{A}_\perp\rangle$$

Question: Why $\phi \to \phi - \vec{v}\cdot\vec{A}$?

Remember Lagrangian for guiding-center dynamics

magnetic fluctuations $\delta\vec{B} = \nabla\times\vec{A}$

$$\delta\vec{B}_\perp \simeq \nabla A_\parallel \times \vec{b} \qquad \delta B_\parallel = \vec{b}\cdot(\nabla\times\vec{A}_\perp)$$

Adiabatic Electron Approximation

electron distribution function

$$f_e = N_e(\psi)\left[\frac{m_e}{2\pi T_e(\psi)}\right]^{3/2} e^{-\frac{mv^2}{2T_e}}\left(1-\frac{e\phi}{T_e}\right) + g_e$$

total electron density $n_e = \int f d\vec{v} = N_e\left(1-\frac{e\phi}{T_e}\right) + \int g_e d\vec{v}$

perturbations to background N_e given by

$$\delta n_e = n_e - N_e = \frac{N_e \phi}{T_e} - \int g_e d\vec{v}$$

equation for $g_e \Rightarrow v_\parallel \nabla_\parallel g_e - i(\omega - \omega_{de}) g_e = \frac{ie}{T_e} J_0 \phi (\omega - \omega_{*e}^T) f_0$

consider $\frac{m_e}{m_i} \ll 1$ such that $k_\parallel v_{the}$ larger than ω, $k_\perp \rho_e$

$v_\parallel \nabla_\parallel g_e$ dominates $\Rightarrow \nabla_\parallel g_e \simeq 0$

$$\Rightarrow \boxed{\delta n_e \simeq \frac{N_e \phi}{T_e}}$$

Close the GK system of equations

equation for ϕ, $\delta B_\parallel$, $\delta \vec{B}_\perp$ using Maxwell's eqs

(1) Poisson's equation $\to$ quasi-neutrality

$\nabla^2 \phi = \frac{e(n_e - n_i)}{\varepsilon_0}$ for small Debye lengths $\lambda_D = \sqrt{\frac{\varepsilon_0 T}{n e^2}}$

$\Rightarrow \nabla^2 \phi \simeq 0 \to n_e \simeq n_i \to$ but $n = \int f d\vec{v}$ not $\int g \, d\vec{v}$!

$$n = \int f d\vec{v} = \int f_0 d\vec{v} + \int g d\vec{v} = N\left(1 - \frac{e\phi}{T}\right) + \int g d\vec{v}$$

assuming $N_e = N_i \Rightarrow \sum_a \frac{N e_a^2}{T_a} \phi = \sum_a e_a \int g_a d\vec{v}$

How to find $\int g\, d\vec{v}$ if $g = g(\vec{R}, \varepsilon, \mu)$, not $g(\vec{r}, \vec{v})$?

Dirac delta trick $\rightarrow$ $\int d\vec{v} = \int \delta(\underbrace{\vec{r} - \vec{r}\,'}_{\vec{R} - \vec{\rho}})\, \underbrace{d\vec{r}\,'\, d\vec{v}\,'}_{\frac{B}{m} d\vec{R}\, dv_\parallel\, d\mu \frac{d\theta}{2\pi}}$

This allows us to keep Poisson's equation in $\vec{r}$ coordinates

$$\int g\, d\vec{v} = \int g\, \delta(\vec{r} - \vec{R} + \vec{\rho}) \frac{B}{m} d\vec{R}\, dv_\parallel\, d\mu \frac{d\theta}{2\pi}$$

$$= \int d\vec{k}\, d\vec{R}\, dv_\parallel\, d\mu \frac{B}{m} g\, \delta(\vec{k})\, e^{i\vec{k}\cdot(\vec{r} - \vec{R})} \underbrace{\int \frac{d\theta}{2\pi} e^{ik_\perp \rho \cos(\theta)}}_{J_0(k_\perp \rho)}$$

$$= \int \underbrace{d\vec{R}\, dv_\parallel\, d\mu \frac{B}{m}}_{= d\vec{v}} J_0(k_\perp \rho)\, g\, \underbrace{\delta(\vec{r} - \vec{R})}_{\text{set } \vec{r} = \vec{R} \text{ in } g}$$

$$= \int J_0(k_\perp \rho)\, g(\vec{r}, \varepsilon, \mu)\, d\vec{v}\, d\vec{k}$$

$\longrightarrow$ Poisson's equation, substituting $\vec{r} \rightarrow \vec{R}$ in Fourier space

$$\boxed{\sum_a \frac{e_a^2 n_a}{T_a} \phi(\vec{R}) = \sum_a e_a \int J_0\, g_a\, d\vec{v}}$$

② Ampère's law $\nabla \times \underset{\boxed{1}}{\delta \vec{B}} = \underset{\boxed{2}}{\mu_0 \vec{j}} + \underset{\boxed{3}}{\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$

displacement current term with $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$

$$\frac{\boxed{3}}{\boxed{1}} \sim \frac{1}{c^2}\frac{\Omega E}{\delta B/L} \sim \frac{\Omega^2 A}{c^2\, A/L^2} \sim \frac{\rho^2 \Omega^2}{c^2} \sim \frac{v_{th}^2}{c^2} \ll 1$$ Non relativistic

using Coulomb gauge $\nabla \cdot \vec{A} = 0 \Rightarrow \nabla \times \delta \vec{B} \simeq -\nabla_\perp^2 A_\parallel \vec{b} + \nabla_\perp \delta B_\parallel \times \vec{b}$

Parallel component: $-\nabla_\perp^2 A_\parallel = \mu_0 \sum_a e_a \int d\vec{v}\, v_\parallel J_0 g_a$

Perpendicular component: $\nabla_\perp \delta B_\parallel = \mu_0 \sum_a e_a \int d\vec{v} \langle \vec{b} \times \vec{v}_\perp g_a \rangle$
introduces J_1

Alternative form of the nonlinear gyrokinetic equation in $(v_\parallel, \mu)$ velocity space coordinates

$$\frac{\partial g}{\partial t} + v_\parallel \nabla_\parallel g + \langle v_\chi \rangle \cdot \nabla g + \vec{v}_d \cdot \nabla g - \mu \nabla_\parallel B \frac{\partial g}{\partial v_\parallel} = \frac{\partial \langle \chi \rangle}{\partial t} f_0 - \langle v_\chi \rangle \cdot \nabla f_0 + \langle C \rangle$$

Poisson bracket formulation of $\langle \vec{V}_E \rangle \cdot \nabla g$

$$\langle \vec{V}_E \rangle = \vec{b} \times \nabla(g_0 \phi) \rightarrow \langle \vec{V}_E \rangle \cdot \nabla g = \vec{b} \times \nabla(g_0 \phi) \cdot \nabla g$$

in an orthogonal coordinate system (x, y, z)
with $\vec{b} = \hat{z}$, $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$

$$\Rightarrow \langle \vec{V}_E \rangle \cdot \nabla g = \frac{\partial \phi}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial g}{\partial x} = \{\phi, g\}$$

(in more complicated geometries there might be coefficients)

Magnetic Geometry and Choice of Coordinates

Use Clebsch representation for $\vec{B}$ (always valid)

$$\boxed{\begin{array}{c} \vec{B} = \nabla \psi \times \nabla \alpha \\ \Rightarrow \vec{k}_\perp = k_\alpha \nabla \alpha + k_\psi \nabla \psi \end{array}}$$

$B = B(\psi, \alpha, z)$

field line following coordinate

where does geometry come in?

8 terms

① B, ② $(\vec{b}\cdot\nabla)z$, ③ $|\nabla\psi|^2$, ④ $|\nabla\alpha|^2$, ⑤ $\nabla\psi\cdot\nabla\alpha$

⑥ $(\vec{b}\times\nabla B)\cdot\nabla\alpha$, ⑦ $(\vec{b}\times\nabla B)\cdot\nabla\psi$ (equal to $(\vec{b}\times\vec{\kappa})\cdot\nabla\psi$), ⑧ $(\vec{b}\times\vec{\kappa})\cdot\nabla\alpha$

example: $\vec{v}_d\cdot\nabla g \longrightarrow \left(\vec{b}\times\nabla\ln B\right)\frac{v_\perp^2}{2\Omega}\cdot\left(\nabla\alpha\frac{\partial g}{\partial\alpha}+\nabla\psi\frac{\partial g}{\partial\psi}+\nabla z\frac{\partial g}{\partial z}\right)$

$(\vec{b}\times\nabla B)\cdot\nabla\alpha$ term

in straight field line coordinates (ψ,θ,ϕ) $\longrightarrow$ $\alpha=\theta-\iota\phi$

$\hookrightarrow$ rotational transform

$\psi=\int dV\,\vec{B}\cdot\nabla\phi$ (toroidal flux)

What is a flux tube?

Minimize computational cost using less grid points, only simulating a flux tube (given location and slope)

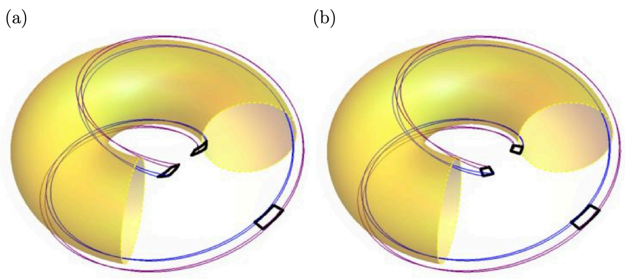
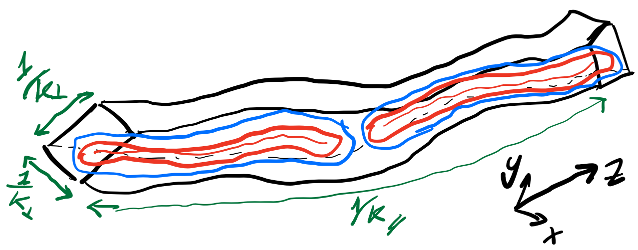


Figure 1. The boundaries of two local simulation domains (thin blue and purple lines): (a) a conventional flux tube and (b) a non-twisting flux tube. Both are one poloidal turn long. Note that at the outboard midplane, both flux tubes have a rectangular cross-section (thick black). However, away from the outboard midplane the cross-section of the conventional flux tube is twisted into a parallelogram, while the non-twisting flux tube remains rectangular. Also shown is the central flux surface of the flux tube (transparent yellow) with a toroidal wedge removed for visual clarity.

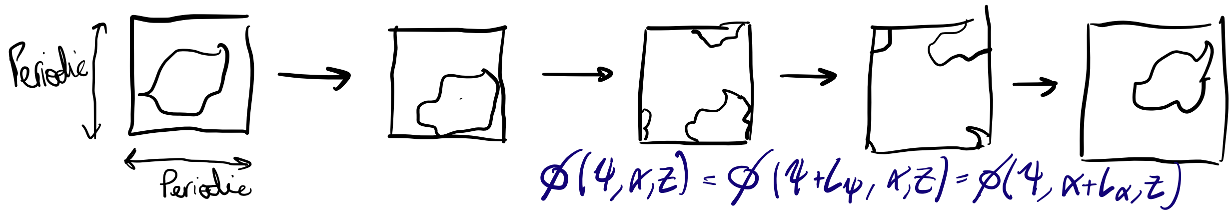
Ball & Brunner, arXiv 2012.04705

Turbulence is anisotropic $\frac{k_\parallel}{k_\perp} \ll 1$



Reduce cost by $\rho_*^2 \sim 10^{-4}$

Assume "Statistical Periodicity" in x, y i.e., (ψ, α)



$$\phi(\psi,\alpha,z) = \phi(\psi + L_\psi, \alpha, z) = \phi(\psi, \alpha + L_\alpha, z)$$

This allows us to work in Fourier space and implement $J_0(k_\perp \rho)$ more accurately

Parallel direction: Long enough flux tube so that "turbulence can't see the edge of the box"

- Twist-and-shift boundary condition

in tokamaks, fluctuations should be similar for poloidal angles $\theta \to \theta + 2\pi N$, at every angle ζ

$\phi(\theta + 2\pi N) = \phi(\theta)$, as $\alpha = \zeta - q\theta$

in Fourier space $\phi \sim e^{ik_\alpha \nabla\alpha + ik_\psi \nabla\psi}$

leading to $[k_\alpha]_{z=\pi N} = [k_\alpha]_{z=-\pi N}$

$$[k_\psi]_{z=\pi N} - [k_\psi]_{z=-\pi N} = 2\pi N q'(\psi) k_\psi$$

quantization on the perpendicular aspect ratio of the domain ← magnetic shear influences boundary conditions

$$\frac{L_x}{L_y} = \frac{\text{integer } \ell}{2\pi N |\hat{s}|}$$

shear $\hat{s} = \frac{\psi}{q}\frac{dq}{d\psi}$

W7-X at $s = \psi/\psi_N = 0.35$, $\hat{s} = 0.019 \to \frac{L_x}{L_y} = 8.1\,\ell$

(Not Confuse with local magnetic shear $s = \vec{B}\cdot\nabla\left(\frac{\nabla\psi\cdot\nabla\alpha}{|\nabla\psi|^2}\right)$)

- Improved boundary condition in M. Martin et al, arXiv 1803.09049

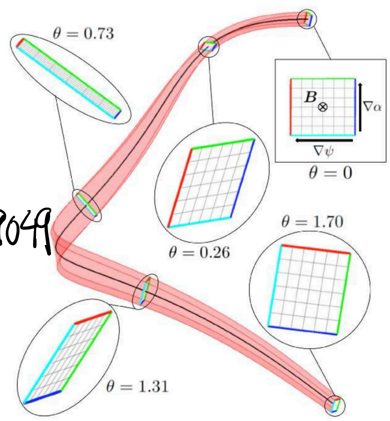


FIG. 5: (Color online) 3D visualization of the $\alpha = 0$ W7-X flux tube in the field-line-following coordinates (ψ, α, z)

FIG. 6: (Color online) 3D visualization of the $\alpha = 0$ flux tube domain in real space superimposed on the flux *surface*

Normalized Coordinates (x, y) measuring the distance from the field line at the centre of the flux tube (ψ_0, α_0)

- $\boxed{x = \frac{dx}{d\psi}(\psi - \psi_0)}$ GS2 code $= L_{ref}(s - s_0)$
- $\boxed{y = \frac{dy}{d\alpha}(\alpha - \alpha_0)}$ GS2 code $= L_{ref}\sqrt{s}(\alpha - \alpha_0)$

constant on a field line

$s = \psi / \psi_N$

• equilibrium quantities

$$N(\psi) = N(\psi_0) + \underbrace{\frac{dN}{d\psi}}_{\text{constant}}(\psi - \psi_0)$$

Quantities from VMEC → equilibrium code, similar to DESC

- cylindrical toroidal angle ζ
- poloidal angle θ_v ⟶ $\theta_p = \theta_v + \Lambda$
- straight field line poloidal angle θ_p (PEST)

field line label
$\alpha = \theta_v + \Lambda - \iota\zeta$

VMEC outputs as a function of (s, θ_v, ζ) → $\Lambda, B, B^\theta = \vec{B}\cdot\nabla\theta_v, B^\zeta = \vec{B}\cdot\nabla\zeta, B_s = \vec{B}\cdot\frac{\partial\vec{r}}{\partial s}, B_\theta = \vec{B}\cdot\frac{\partial\vec{r}}{\partial\theta_v}, B_\zeta = \vec{B}\cdot\frac{\partial\vec{r}}{\partial\zeta}$

why not B^ψ

jacobian $\sqrt{g} = \frac{\partial\vec{r}}{\partial s}\cdot\frac{\partial\vec{r}}{\partial\theta_v}\times\frac{\partial\vec{r}}{\partial\zeta} = \frac{1}{\nabla s\cdot(\nabla\theta_v\times\nabla\zeta)}$

Take parallel coordinate z angle ζ → $\text{gradpar} = L_{ref}\frac{B^\zeta}{B}$

B^ζ → Bsupvmnc

$(\vec{B} \times \nabla B) \cdot \nabla \psi$ using $B_\theta, B_\zeta, B, \sqrt{g}$

gbdrift0, cvdrift0

$$\nabla B = \frac{\partial B}{\partial s}\nabla s + \frac{\partial B}{\partial \theta_v}\nabla\theta_v + \frac{\partial B}{\partial \zeta}\nabla\zeta$$

$$\vec{B} = B_s \nabla s + B_{\theta_v}\nabla\theta_v + B_\zeta \nabla\zeta$$

$$\vec{B} \times \nabla B \cdot \nabla\psi = \frac{d\psi}{ds}\left(B_{\theta_v}\nabla\theta_v \times \nabla\zeta \cdot \nabla s \frac{\partial B}{\partial \zeta} + B_\zeta \nabla\zeta \times \nabla\theta_v \cdot \nabla s \frac{\partial B}{\partial \theta_v}\right)$$

$$= \frac{\psi_{LCFS}}{\sqrt{g}}\left(B_\theta \frac{\partial B}{\partial \zeta} - B_\zeta \frac{\partial B}{\partial \theta_v}\right)$$

same for other components

How do we numerically solve the GK equation?

Simple form of GK equation

$$\frac{\partial g}{\partial t} + A\frac{\partial g}{\partial z} + Bg = C\frac{\partial(g_0\phi)}{\partial t} + E(g_0\phi) \quad \text{(linearized)}$$

$\frac{\partial g}{\partial t} \sim k_\parallel v_\parallel$; $A\frac{\partial g}{\partial z} \sim \omega_D \sim k_y\rho$; $C\frac{\partial(g_0\phi)}{\partial t} \sim \frac{1}{T}$; $E(g_0\phi) \sim \omega^{*T} \sim k_\perp\rho$

$$g = g(z, k_x, k_y, \varepsilon, v_\parallel) \Rightarrow \text{Discretization on every coordinate}$$

stored on a grid: ntheta × nkx × nky × nε × nλ

discretized $\varphi_{i \to \text{space}}^{n \to \text{time}} \to \frac{\partial\varphi}{\partial t} \sim \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t}$; $\frac{\partial\varphi}{\partial z} \sim \frac{\varphi_{i+1}^n - \varphi_i^n}{\Delta z}$

GK eq $\rightarrow \underline{\underline{A}}\, g^{n+1} + \underline{\underline{B}}\, g^{n} = \underline{\underline{D}}\, \varphi^{n+1} + \underline{\underline{E}}\, \varphi^{n}$

Quasineutrality $\rightarrow \underline{\underline{F}}\, \varphi^{n+1} = \underline{\underline{G}}\, g^{n+1}$

Matrix Multiplication $\rightarrow$

$$\boxed{\begin{aligned} g^{n+1} &= \underline{\underline{M}}^{-1}\left(\underline{\underline{E}}\, \varphi^{n} - \underline{\underline{B}}\, g^{n}\right) \\ \underline{\underline{M}} &= \left(\underline{\underline{A}} - \underline{\underline{D}}\, \underline{\underline{F}}^{-1} \underline{\underline{G}}\right) \end{aligned}}$$

solution found!!

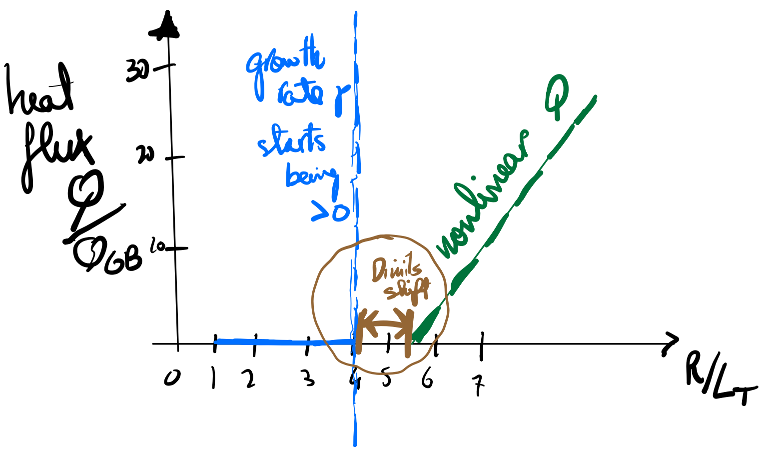
Nonlinearity

- treated explicitly with a CFL condition to keep the step size small enough for stability
- sum of Fourier components of φ and g

Interesting fact from nonlinear dynamics

Dimits Shift

- linear critical gradient (without zonal flows)
- turbulent critical gradient (with zonal flows)



(if there's time)

Gyrofluid Approach to Gyrokinetics

(gyro-Landau fluid)

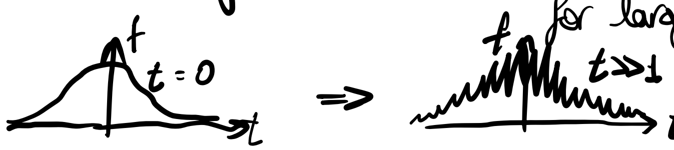
can be defined as an n-pole approximation to the plasma distribution function

expand f in Hermite polynomials: $f(x,v,t) = \dfrac{e^{-v^2/v_{th}^2}}{\sqrt{\pi}\, v_{th}} \sum_{\ell=0}^{\infty} \dfrac{N^{\ell}_{(x,t)}}{2^{\ell}\ell!} H_\ell\left(\dfrac{v}{v_{th}}\right)$

$H_\ell(v) = (-1)^\ell e^{v^2} \dfrac{d^\ell}{dv^\ell}\left(e^{-v^2}\right)$

example: ① $\dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} = 0 \Rightarrow f = f_M(v)\, e^{ik(x-vt)}$ called free streaming

because $f \sim \sin(kvt) \Rightarrow$ it becomes extremely oscillatory in v for large t



needs many Hermite polynomials

density decay via phase-mixing $\Big\}$ $n = \int f dv = e^{ikx}\, \boxed{e^{-k^2t^2/4}} \rightarrow 0$

energy flows to higher moments

① with Hermite $\Rightarrow \dfrac{\partial N^{l}}{\partial t} + v_{th}\left(l\,\dfrac{\partial N^{l-1}}{\partial x} + \dfrac{1}{2}\dfrac{\partial N^{l+1}}{\partial x}\right) = 0$

- infinite set of coupled equations
- closure approximations needed

take $\partial/\partial t$, Fourier transform x, normalize time to $\dfrac{1}{k v_{th}}$

$$\Rightarrow \frac{\partial^2 N^l}{\partial t^2} + l(l-1)N^{l-2} + \left(l+\tfrac{1}{2}\right)N^l + \frac{N^{l+2}}{4} = 0$$

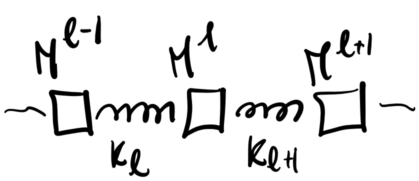
scale $N^{2l} = (-1)^l\, 2^l\, (2l-1)!!\, \bar{N}^l(t)$, $\quad l!! = l(l-2)(l-4)\ldots 3\cdot 1$

Infinite set of spring masses coupled to their nearest neighbors with mass M_{l-1} and M_{l+1} and spring constant K_l and K_{l+1}

$$\boxed{M_l \frac{d^2\bar{N}^l}{dt^2} = K_{l+1}\left(\bar{N}^{l+1} - \bar{N}^l\right) - K_l\left(\bar{N}^l - \bar{N}^{l-1}\right)}$$

$M_l = \dfrac{(2l)!}{4^l (l!)^2} \sim \dfrac{1}{\sqrt{\pi l}}$, $\quad K_l = l M_l \sim \sqrt{\dfrac{l}{\pi}}$ $\quad$ speed of oscillation $= \sqrt{\dfrac{K}{m}} = \sqrt{l}$

analytic solution $N^l = \displaystyle\int_{-\infty}^{\infty} f H_l e^{-v^2}\, dv = t^l e^{-(k v_{th} t)^2/2}$



initial density perturbation

$\downarrow$

perturbation of the first mass of the system

Truncation $\bar{N}^{l_{MAX}} = 0 \rightarrow$ fixed wall reflecting the energy back to lower moments

wave reaches l^{th} moment ($\frac{l}{2}$ mass) at $t = \sqrt{l}/k v_{th}$

Need damping before reaching the wall

$\Downarrow$

collision operator $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = C(f) = \nu\frac{\partial}{\partial v}\left(vf + v_{th}^2\frac{\partial f}{\partial v}\right)$

$$\Rightarrow \frac{\partial N^\ell}{\partial t} + v_{th}\left(\ell\frac{\partial N^{\ell-1}}{\partial x} + \frac{1}{2}\frac{\partial N^{\ell+1}}{\partial x}\right) = -\nu\ell N^\ell$$

damping increases with ℓ

Phase-mixing competing with driving f back to a Maxwellian

Alternative : Hammet-Perkins closure

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}E\frac{\partial f}{\partial v} = 0 \xrightarrow[\text{linearize}]{\text{Fourier-Laplace}} \delta f = \frac{e\,\delta\phi}{m}\frac{\partial f_0/\partial v}{v-\omega/k}$$

consider $f_0 = n_0\frac{e^{-v^2/v_{th}^2}}{\sqrt{\pi}\,v_{th}}$

$$\delta N^3 = \int \delta f\, H_3(v)e^{-v^2}dv \sim A_1\,\delta\phi$$

$$\delta N^2 = \int \delta f\, H_2(v)e^{-v^2}dv \sim A_2\,\delta\phi$$

$\Downarrow$

$$\boxed{\delta N^3 = \frac{A_1}{A_2}\delta N^2}$$

$$\delta Q = -i\, v_{th}\, n_0 v_{th}\, \delta T\, \chi\left(\tfrac{\omega}{k v_{th}}\right) \xleftarrow[\text{and temperature}]{\text{for Heat flux}}$$

similar to plasma dispersion function $\chi(x) = i\frac{(2x^3-3x)Z(x)+2x^2-2}{(2x^2-1)Z(x)+2x}$, $Z(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-x}dx$, $Z(0) = i\sqrt{\pi}$

Gyrofluid equations $\frac{d}{dt}N^\ell = S(N^\ell, N^{\ell+1}, N^{\ell-1}, \phi)$ //