MAX-PLANCK-INSTITUT
FÜR PLASMAPHYSIK

Introduction to gyrokinetics

- a quick and dirty derivation

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EUROfusion

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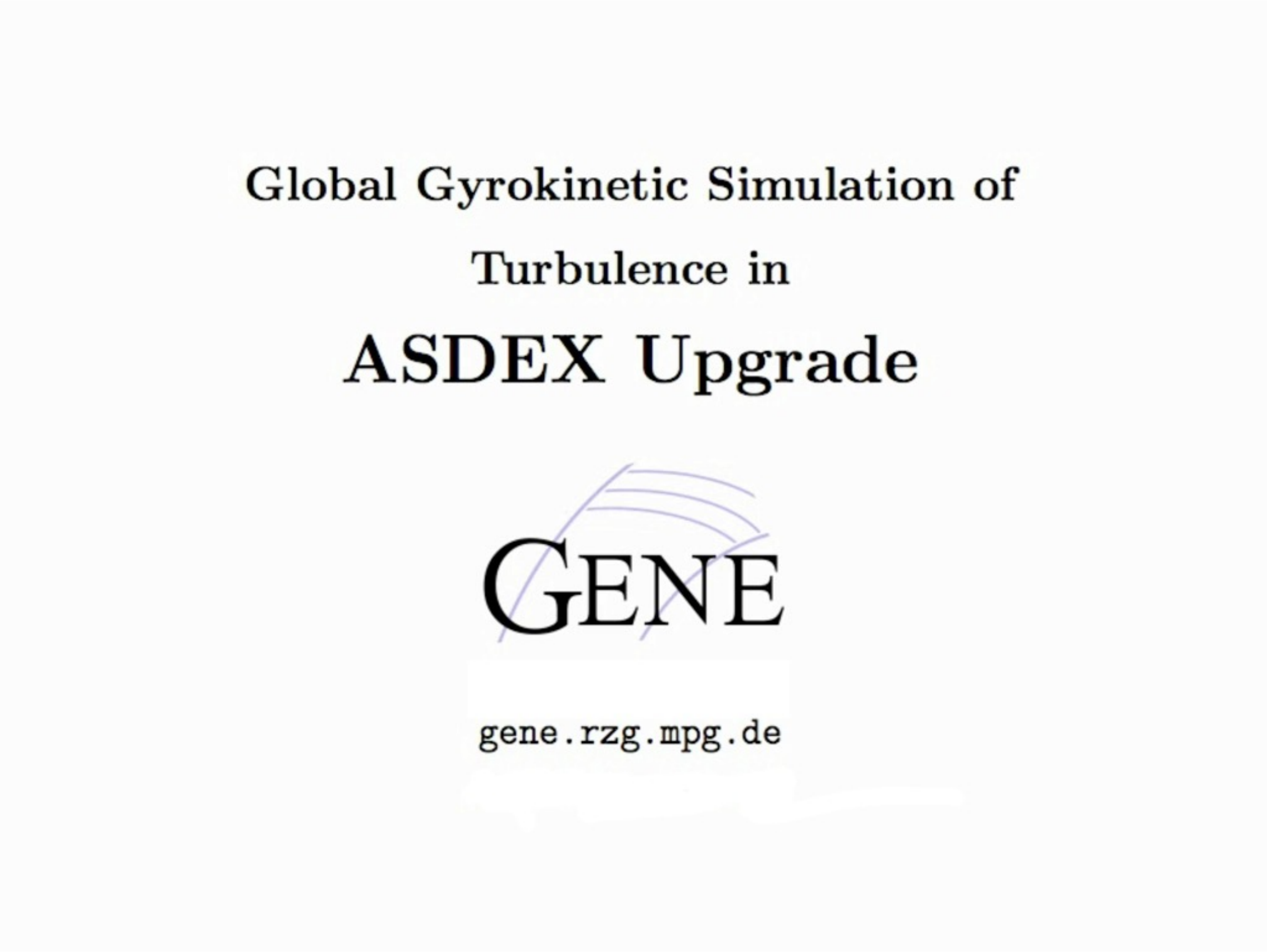
Gradients in plasma lead to unwanted turbulence



Global Gyrokinetic Simulation of Turbulence in

ASDEX Upgrade

GENE

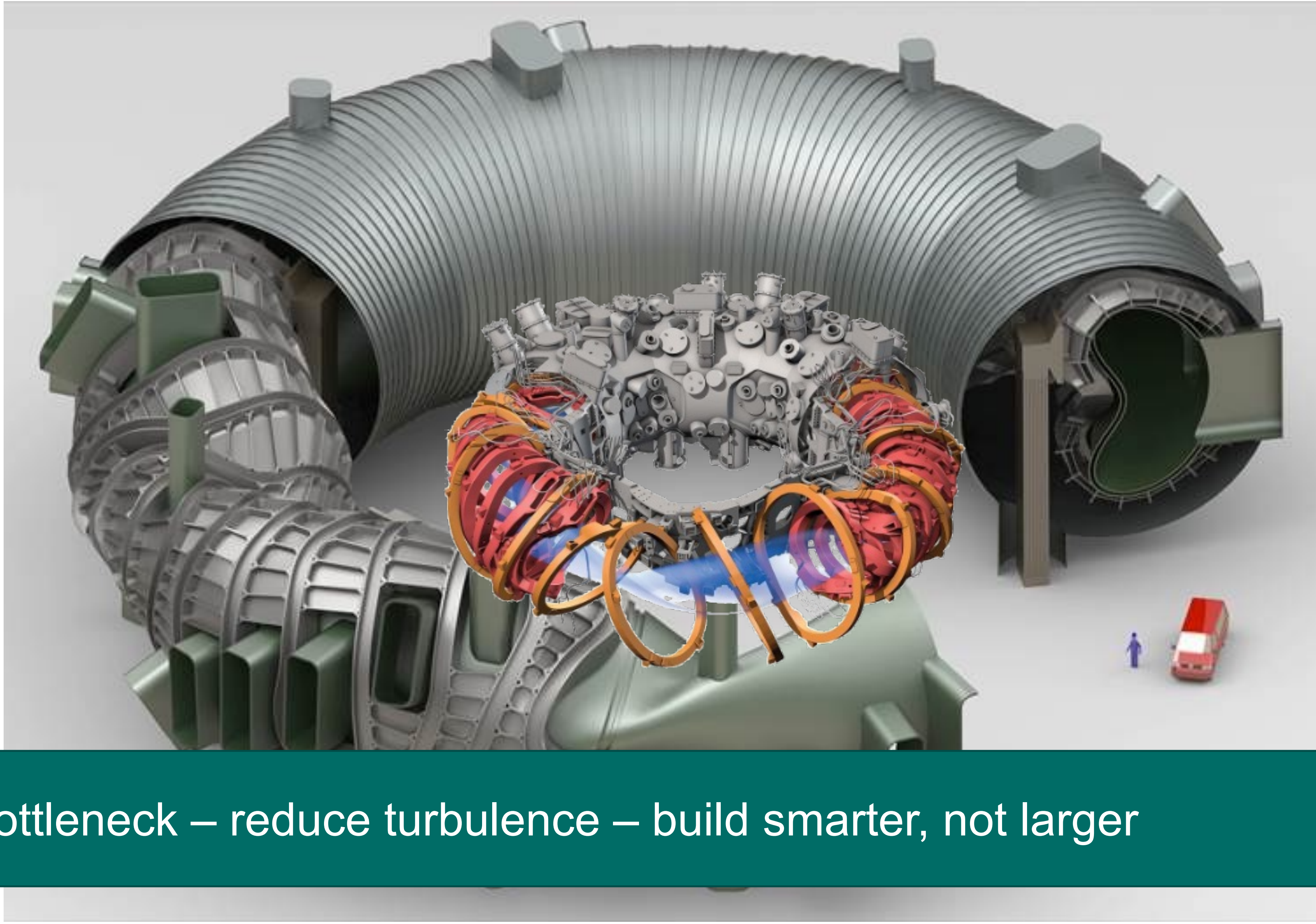
gene.rzg.mpg.de

Density and temperature gradients:

Turbulence!

Wendelstein 7-X:
1Bn EUR

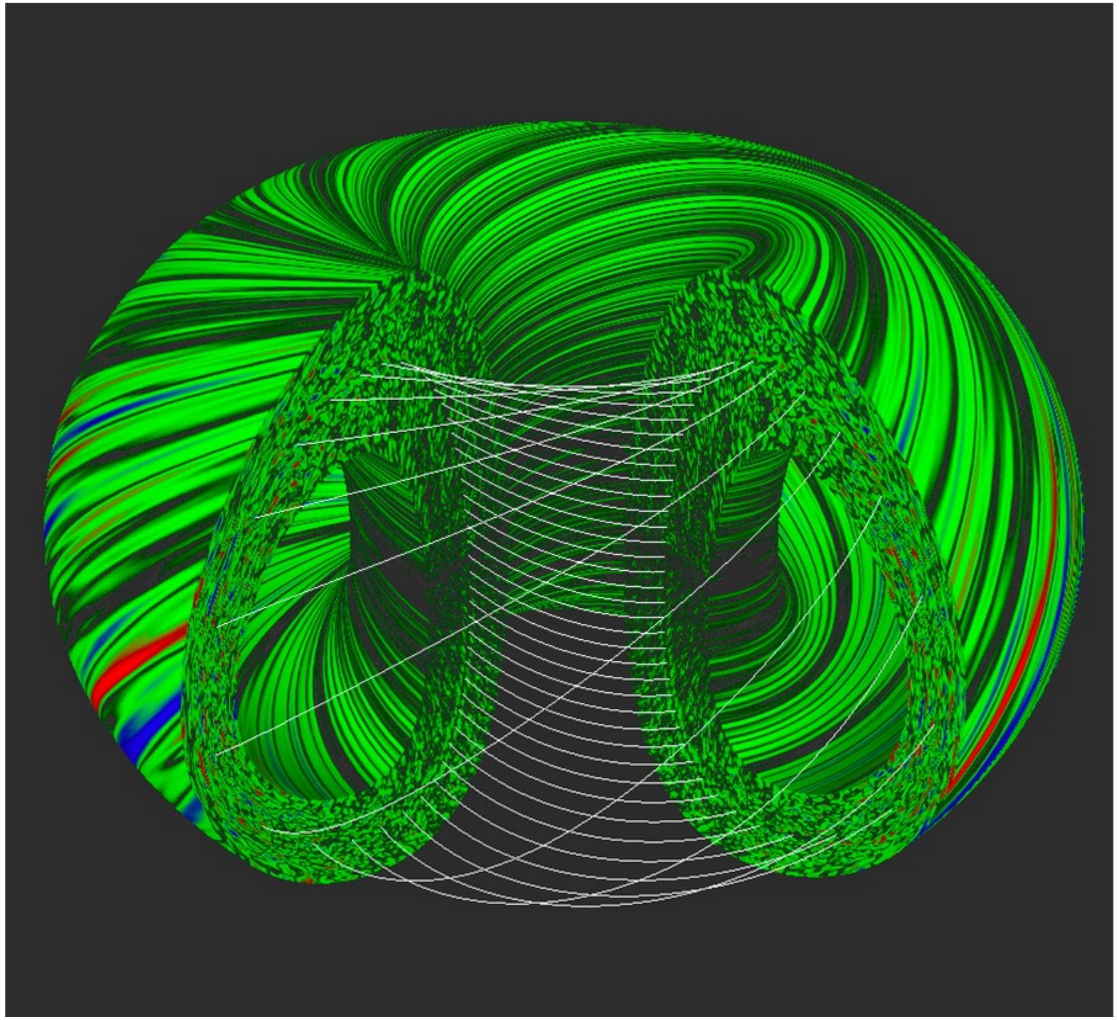
**Reactor-sized:
20 Bn EUR**



Turbulence as bottleneck – reduce turbulence – build smarter, not larger

Fluctuations in fusion plasmas

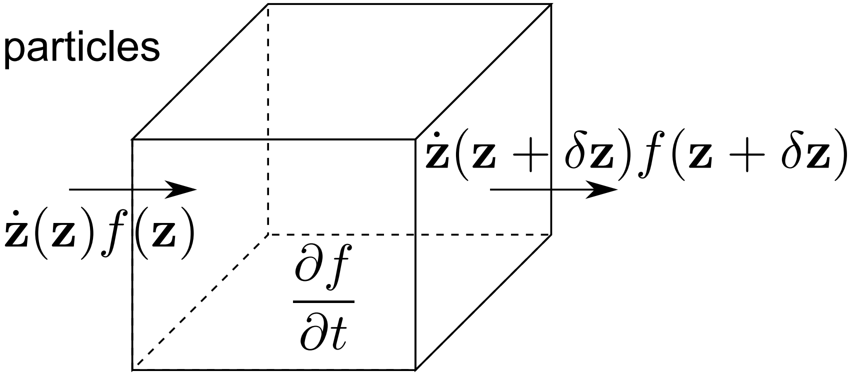
- Observed features:
- Everything fluctuates
  (n, Te, Ti, electromagnetic fields)
- Fluctuation levels tend to be small – $\frac{\tilde{n}}{n} \approx 0.1 \ldots 1\%$
- Fluctuation frequencies: 10 – 1000s kHz
  are smaller than the gyrofrequencies
- Perpendicular fluctuation scales much smaller than the system size – on the order of the gyroradius
- Fluctuations extended along the field lines



Can’t use MHD,
but allows the use of gyrokinetics!

The kinetic equation – a continuity equation of the distribution in 6D phase space

- Kinetic description – intermediate step between fluid description and resolving single-particle dynamics
- Describe evolution of particles in terms of distribution function f_a of particles in 6D phase space **z** (3D real space **x**, 3D velocity space **v**)



$$\frac{\partial f_a}{\partial t} + \nabla_{\mathbf{z}}\left(\dot{\mathbf{z}} f_a\right) = C_a(f_a). \qquad \dot{\mathbf{z}} = \left(\dot{\mathbf{x}}, \dot{\mathbf{v}}\right)$$

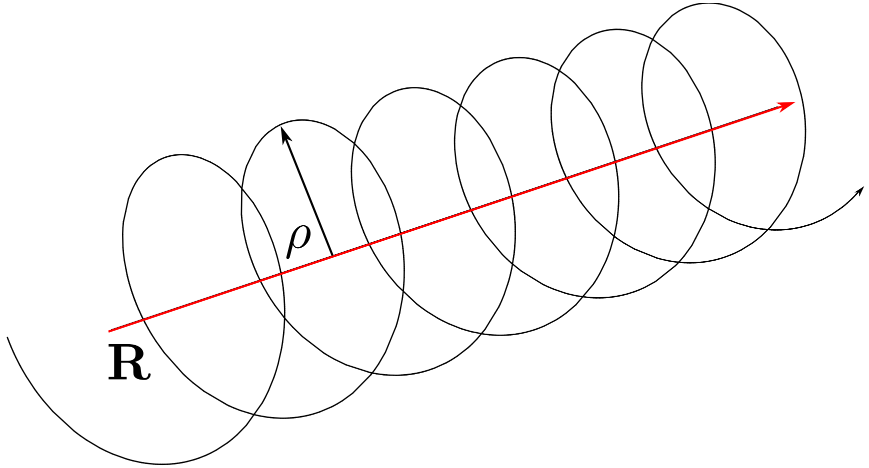
$$\dot{\mathbf{v}} = \frac{e_a}{m_a}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \quad \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0 \quad \nabla_{\mathbf{v}} \cdot \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = 0$$

$$\boxed{\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_a + \frac{e_a}{m_a}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}} f_a = C_a(f_a)}$$

The kinetic equation… now in more convenient coordinates

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_a + \frac{e_a}{m_a} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_a = C_a(f_a)$$

- Freedom in choosing suitable coordinates for **x** and **v**
- Make use of coordinates related to the gyromotion:
  - Gyro angle ϑ
  - Energy $\mathcal{E} = \frac{m_a v^2}{2} + e_a \phi$
  - Magnetic moment $\mu = \frac{m_a v_\perp^2}{2B}$
  - Gyro-centre $\mathbf{R} = \mathbf{r} + \frac{\hat{\mathbf{b}} \times \mathbf{v}}{\Omega_a}$



$$\frac{\partial f_a}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial f_a}{\partial \mathbf{R}} + \dot{\vartheta} \frac{\partial f_a}{\partial \vartheta} + \dot{\mathcal{E}} \frac{\partial f_a}{\partial \mathcal{E}} + \dot{\mu} \frac{\partial f_a}{\partial \mu} = C_a$$

The kinetic equation in more convenient coordinates cont'd

$$\boxed{\frac{\partial f_a}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial f_a}{\partial \mathbf{R}} + \dot{\vartheta}\frac{\partial f_a}{\partial \vartheta} + \dot{\mathcal{E}}\frac{\partial f_a}{\partial \mathcal{E}} + \dot{\mu}\frac{\partial f_a}{\partial \mu} = C_a}$$

We can simplify:

$$m_a \dot{\mathbf{v}} = e_a \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

$$\dot{\vartheta} \simeq -\Omega_a$$

$$\begin{aligned}
\dot{\mathcal{E}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_a v^2}{2}\right) + e_a\left(\frac{\partial \phi}{\partial t}_{\mathbf{r}}\right) + e_a \frac{\partial \phi}{\partial \mathbf{r}} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\
&= m_a \dot{\mathbf{v}} \cdot \mathbf{v} + e_a\left(\frac{\partial \phi}{\partial t}_{\mathbf{r}}\right) + e_a \mathbf{v} \cdot \nabla \phi \\
&= -e_a \mathbf{v} \cdot \nabla \phi + e_a\left(\frac{\partial \phi}{\partial t}_{\mathbf{r}}\right) + e_a \mathbf{v} \cdot \nabla \phi \\
&= e_a \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{r}}
\end{aligned}$$

Simplifying the kinetic equation some more

We decompose the distribution function $f_a = f_{a0} + g_a,$

f_{a0} equilibrium function including adiabatic responses to small electric fields $\propto -e_a\phi/T_a,$

g_a Small non-adiabatic part with $g_a \ll f_{a0}$

Assumptions:

- Equilibrium varies slowly in time c.f. perturbed part $\frac{\partial g_a}{\partial t} \gtrsim \frac{\partial f_{a0}}{\partial t}$
- But allow $\nabla g_a \sim \nabla f_{a0}.$

- And further: small perturbations in electrostatic field, equilibrium length scales L large compared with gyroradius, frequencies of instabilities w small compared with gyrofrequency:

$$\frac{e_a\phi}{T_a} \sim \frac{\rho}{L} \sim \frac{\omega}{\Omega_a} \sim \delta \ll 1$$

Your turn: ordering of the terms in δ

$$\boxed{\frac{\partial f_a}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial f_a}{\partial \mathbf{R}} + \dot{\vartheta} \frac{\partial f_a}{\partial \vartheta} + \dot{\mathcal{E}} \frac{\partial f_a}{\partial \mathcal{E}} + \dot{\mu} \frac{\partial f_a}{\partial \mu} = C_a}$$

$$\dot{\vartheta} \simeq -\Omega_a \qquad \dot{\mathcal{E}} = e_a \left(\frac{\partial \phi}{\partial t} \right)_{\mathbf{r}}$$

$$f_a = f_{a0} + g_a, \qquad g_a \ll f_{a0} \qquad \nabla g_a \sim \nabla f_{a0}, \qquad \frac{\partial g_a}{\partial t} \gtrsim \frac{\partial f_{a0}}{\partial t}$$

$$\frac{\partial g_a}{\partial \mathcal{E}} \ll \frac{\partial f_{a0}}{\partial \mathcal{E}}$$

$$\frac{\partial g_a}{\partial \mu} \ll \frac{\partial f_{a0}}{\partial \mu}$$

In lowest orders

Equilibrium is independent of gyroangle: $\partial f_{a0}/\partial\vartheta = 0$

In next order $$\frac{\partial f_{a0}}{\partial t} + \frac{\partial g_a}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}}(f_{a0} + g_a) - \Omega_a \frac{\partial g_a}{\partial \vartheta} + e\frac{\partial \phi}{\partial t}\frac{\partial f_{a0}}{\partial \mathcal{E}} + \dot{\mu}\frac{\partial f_{a0}}{\partial \mu} = C_a,$$

Now choosing the equilibrium to be represented by a Maxwellian

(follows from drift kinetics in lowest order, at high collisionality)

$$f_{a0} = n_a(\psi)\left(\frac{m_a}{2\pi T_a(\psi)}\right)^{3/2} e^{-\mathcal{E}/T_a(\psi)}$$

$$\simeq n_a(\psi)\left(\frac{m_a}{2\pi T_a(\psi)}\right)^{3/2} e^{-m_a v^2/2T_a(\psi)}\left(1-\frac{e_a\phi}{T_a}\right)$$

- Note:
  density n and temperature T are assumed to be flux quantities with flux label ψ

- Then:

$$\frac{\partial f_{a0}}{\partial t} = 0$$

$$\frac{\partial f_{a0}}{\partial \mu} = 0$$

$$\frac{\partial f_{a0}}{\partial \mathcal{E}} = -\frac{f_{a0}}{T_a}$$

Splitting the movement into parallel and gyro motion

Assuming the particle mostly moves along the field $\dot{\mathbf{R}} = v_{\|}\hat{\mathbf{b}} + \mathcal{O}(\delta v_T)$

We obtain $$\frac{\partial g_a}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}}(f_{a0} + g_a) - \Omega \frac{\partial g_a}{\partial \vartheta} - \frac{e}{T}\left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{r}} f_{a0} = C_a.$$

All terms are of order $\mathcal{O}(\delta \omega f_{a0})$ with $\omega \sim v_T/L,$ except $\Omega \frac{\partial g_a}{\partial \vartheta} \sim \omega f_{a0}$ as largest term

We can thus expand $g_a = g_{a0} + g_{a1} + \ldots$ yielding in lowest order $\Omega \frac{\partial g_{a0}}{\partial \vartheta} = 0$

In next order after gyroaveraging hile keeping the gyro centre constant

$$\frac{\partial g_{a0}}{\partial t} + \langle \dot{\mathbf{R}} \rangle_{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}}(f_{a0} + g_{a0}) - \frac{e_a}{T_a}\left\langle \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{r}} \right\rangle_{\mathbf{R}} f_{a0} = \langle C_a \rangle_{\mathbf{R}}.$$

Averaging out the fast gyromotion

- Magnetic field → reduction to 5D phase space

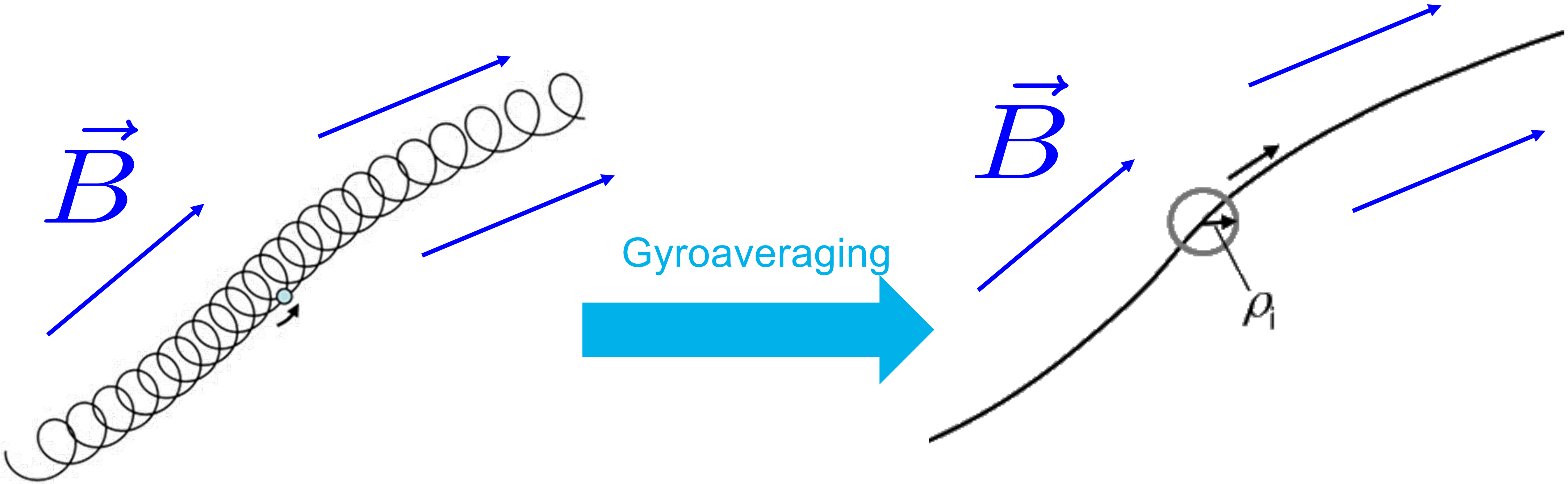


Image source: Garbet *et al. Nucl. Fusion* (2010).

Splitting the velocity

$$\mathbf{v}_E = \frac{\hat{\mathbf{b}} \times \nabla \langle \phi \rangle_{\mathbf{R}}}{B}$$

Splitting the velocity: $\langle \dot{\mathbf{R}} \rangle_{\mathbf{R}} = \frac{1}{2\pi} \oint \dot{\mathbf{R}} \, \mathrm{d}\vartheta = v_{\|} \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d$

$$\mathbf{v}_{da} = \frac{\hat{\mathbf{b}}}{\Omega_a} \times \left(\frac{v_\perp^2}{2} \nabla \ln B + v_\|^2 \boldsymbol{\kappa} \right)$$

Using that f_{a0} is a flux function and thus does not vary along the field lines $\mathbf{b} \cdot \nabla f_{a0} = 0$

$$\frac{\partial g_{a0}}{\partial t} + \left(v_\| \hat{\mathbf{b}} + \underbrace{\mathbf{v}_E}_{A} + \mathbf{v}_d \right) \cdot \nabla g_{a0} - \langle C \rangle_{\mathbf{R}} = - \left(\mathbf{v}_E + \underbrace{\mathbf{v}_d}_{B} \right) \cdot \nabla f_{a0} + \frac{e}{T} \left\langle \left(\frac{\partial \phi}{\partial t} \right)_{\mathbf{r}} \right\rangle_{\mathbf{R}} f_{a0}$$

nonlinear

Neoclassical response

Executing the gyroaverages: $\frac{e}{T}\left\langle\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}}\right\rangle_{\mathbf{R}} f_{a0}$

Slowly varying

Have slow variation along the field but fast across it, can write: $\phi(\mathbf{r},t) = \hat{\phi}(\mathbf{r},\omega)e^{\underbrace{i(S(\mathbf{r})/\delta-\omega t)}_{\text{Fast variation}}}$

Define $\mathbf{k}_\perp = \frac{\nabla S}{\delta}$ with $k_\perp\rho = \mathcal{O}(1)$

Ballooning transform allows assumption of $\nabla_\| S = 0.$

Then $\left\langle\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}}\right\rangle_{\mathbf{R}} = -i\omega\left\langle\phi(\mathbf{R}+\boldsymbol{\rho})\right\rangle_{\mathbf{R}} \simeq -i\omega\hat{\phi}(\mathbf{R})e^{-i\omega t}\left\langle e^{iS(\mathbf{R}+\boldsymbol{\rho})/\delta}\right\rangle_{\mathbf{R}}$

where $\left\langle e^{iS(\mathbf{R}+\boldsymbol{\rho})/\delta}\right\rangle_{\mathbf{R}} \simeq e^{iS(\mathbf{R}/\delta)}\left\langle e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}\right\rangle_{\mathbf{R}} = J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)e^{iS(\mathbf{R}/\delta)}$

So that $\boxed{\left\langle\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}}\right\rangle_{\mathbf{R}} = -i\omega J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\phi(\mathbf{R},t)}$

Bessel function of zeroth order

$$\int_0^{2\pi} e^{ix\sin\vartheta}\,\mathrm{d}\vartheta = \int_0^{2\pi}\cos(x\sin\vartheta)\,\mathrm{d}\vartheta = 2\pi J_0(x)$$

Executing the gyroaverages

$$-\mathbf{v}_E \cdot \nabla f_{a0} = -\frac{\hat{\mathbf{b}} \times \nabla \langle \phi \rangle_{\mathbf{R}}}{B} \cdot \nabla f_{a0} = \frac{\nabla \langle \phi \rangle_{\mathbf{R}}}{B} \cdot \hat{\mathbf{b}} \times \nabla f_{a0}.$$

$$\begin{aligned}
\nabla \langle \phi \rangle_{\mathbf{R}} &= \nabla \langle \phi(\mathbf{R} + \boldsymbol{\rho}) \rangle_{\mathbf{R}} \\
&\simeq \nabla \left[\hat{\phi}(\mathbf{R}) \left\langle e^{iS(\mathbf{R}+\boldsymbol{\rho})/\delta} \right\rangle_{\mathbf{R}} e^{-i\omega t} \right] \\
&\simeq \nabla \left(\hat{\phi}(\mathbf{R}) e^{-i\omega t} e^{iS(\mathbf{R})/\delta} J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \right) \\
&\simeq \underbrace{\hat{\phi}(\mathbf{R}) e^{-i\omega t} e^{iS(\mathbf{R})/\delta}}_{\phi(\mathbf{R},t)} i \frac{\nabla S}{\delta} J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \quad \text{because } \frac{\nabla S}{\delta} \gg \frac{\nabla \hat{\phi}(\mathbf{R})}{\hat{\phi}(\mathbf{R})} \\
&\simeq i \mathbf{k}_\perp J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \phi(\mathbf{R}, t).
\end{aligned}$$

We thus obtain

$$-\mathbf{v}_E \cdot \nabla f_{a0} = i J_0 \left(\frac{k_\perp v_\perp}{\Omega} \right) \phi(\mathbf{R}, t) \frac{1}{B} \mathbf{k}_\perp \cdot \hat{\mathbf{b}} \times \nabla f_{a0}$$

Simplify further with gradients in the distribution function

With f_{a0} being a flux function: $\nabla f_{a0} = \frac{\partial f_{a0}}{\partial \psi} \nabla \psi$

Magnetic field in Clebsch coordinates: $\mathbf{B} = \nabla \psi \times \nabla \alpha$ ➔ $\mathbf{k}_\perp = k_\psi \nabla \psi + k_\alpha \nabla \alpha$

Obtain

$$\begin{aligned} -\mathbf{v}_E \cdot \nabla f_{a0} &= i J_0 \phi \frac{\hat{\mathbf{b}} \cdot (\nabla \psi \times \mathbf{k}_\perp)}{B} \frac{\partial f_{a0}}{\partial \psi} \\ &= i J_0 \phi k_\alpha \left[\frac{d \ln n_a}{d \psi} + \left(\frac{\mathcal{E}}{T_a} - \frac{3}{2} \right) \frac{d \ln T_a}{d \psi} \right] f_{a0} \\ &= i J_0 \frac{e_a \phi}{T_a} \omega_{*a}^T f_{a0} \end{aligned}$$

With $\omega_{*a}^T = \omega_{*a} \left[1 + \eta_a \left(\frac{\mathcal{E}}{T_a} - \frac{3}{2} \right) \right]$ $\omega_{*a} = \frac{T_a k_\alpha}{e_a} \frac{d \ln n_a}{d \psi}$ $\eta_a = \frac{d \ln T_a}{d \psi} \Big/ \frac{d \ln n_a}{d \psi}$

Final step: same separation also in g_{a0}

With $g_{a0}(\mathbf{R},\mathcal{E},\mu,t) = \hat{g}_a(\mathbf{R},\mathcal{E},\mu)e^{i(S(\mathbf{R})/\delta-\omega t)}$

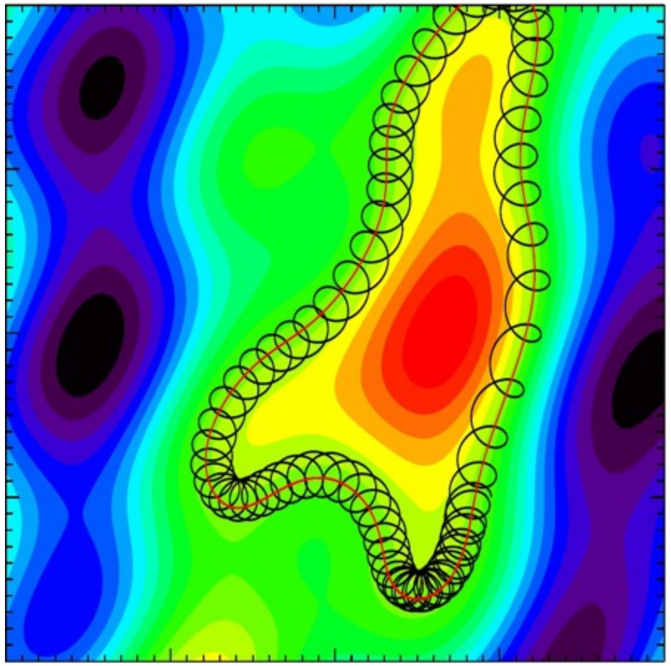
Obtain $\frac{\partial g_{a0}}{\partial t} + \left(v_\parallel \hat{\mathbf{b}} + \mathbf{v}_{da}\right) \cdot \nabla g_{a0} \simeq \left[v_\parallel \nabla_\parallel \hat{g}_a - i(\omega - \omega_{da})\hat{g}_a\right] e^{i(S(\mathbf{R})/\delta-\omega t)}$

With $\omega_{da} = \mathbf{k}_\perp \cdot \mathbf{v}_{da}.$

Leaving out terms A and B (nonlinearity and neoclassical response) and collisions

$$\boxed{v_\parallel \nabla_\parallel g_a - i(\omega - \omega_{da})g_a = -\frac{ie_a}{T_a} J_0\phi \left(\omega - \omega_{*a}^T\right) f_{a0}}$$

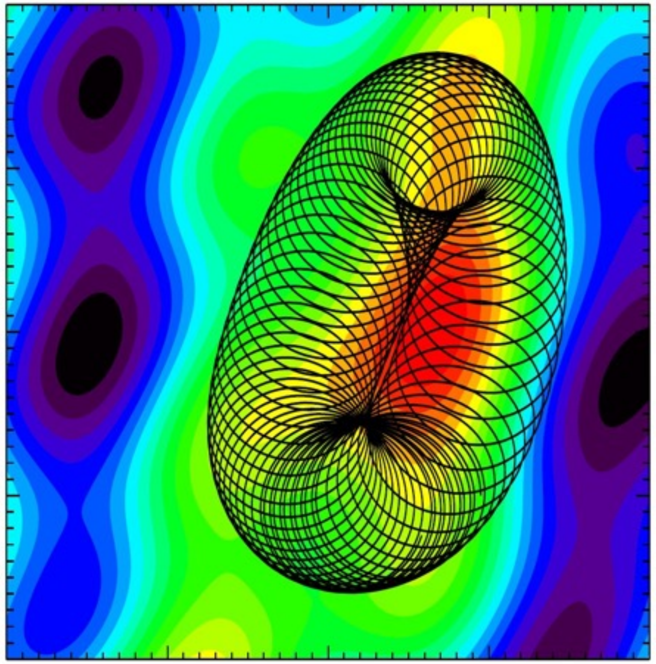
Suitability of gyroaveraging: small gyration



Gyromotion small
compared with background:
gyroaveraging allowed

Important condition:

$$\rho_s \ll L$$



Gyromotion large
compared with background:
gyroaveraging not allowed