Symmetries of magnetic fields:

What are they?

Outline

(I) Intro to differential forms

(II) Pullbacks

(III) Symmetry

(I) Intro to differential forms

<u>Recall</u>: The mapping

$$\overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n\text{-times}} \longrightarrow \mathbb{R}$$

$$(v_1, \ldots, v_n) \longmapsto \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

- gives signed volume of parallelepiped spanned by $v_1, \ldots, v_n$.
- satisfies two key properties:

$$\boxed{\begin{array}{l} \text{① Linear in each } v_k \text{ separately} \\ \\ \text{② Skew symmetric under exchanges } v_j \leftrightarrow v_\ell \end{array}}$$

Q: Can we generalize det

to fewer than n vectors?

(det for rectangular matrices)

Observe: if $u, v \in \mathbb{R}^3$ then

- $\det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}$ doesn't make sense, but...

- The maps (maximal minors)

$$(u,v) \mapsto \det \begin{bmatrix} - & - \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}, \quad (u,v) \mapsto \det \begin{bmatrix} u_1 & v_1 \\ - & - \\ u_3 & v_3 \end{bmatrix}, \quad (u,v) \mapsto \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ - & - \end{bmatrix}$$

* give projected areas in y-z, x-z, and x-y planes
* satisfy properties ① and ② above

$\boxed{\text{Q: how many maximal minors for 3 vectors in } \mathbb{R}^4 \text{ ?}}$

$u, v, w \in \mathbb{R}^4$

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix}$$

Maximal minors

what we keep from det

- properties (1) + (2)
- interpretation as (signed, projected) volume

what we lose from det

- uniqueness (one maximal minor for each possible projection)

Fairly satisfying!

Definition: A k-form γ on $\mathbb{R}^n$ is a mapping

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}} \longrightarrow \mathbb{R}$$

$$(v_1, \ldots, v_k) \longmapsto \gamma(v_1, \ldots, v_k)$$

that satisfies properties ① and ②

$\boxed{\text{Example}}$

with $n = 3$, $B \in \mathbb{R}^3$ a fixed vector, $c \in \mathbb{R}$ a fixed number

$\cdot \longmapsto c$ is a 0-form

$u \longmapsto B \cdot u$ is a 1-form

$u, v \longmapsto B \cdot u \times v$ is a 2-form

$u, v, w \longmapsto u \cdot v \times w$ is a 3-form

Proposition:

(A) the set of all K-forms over $\mathbb{R}^n$ is a vector space Λ^k

(B) if $k > n$ then the only k-form is 0.

(C) if $k \le n$ then the collection of maximal minors is a k-form basis.

e.g. in $\mathbb{R}^3$ the 2-form from previous slide is:

$$(u,v) \mapsto B \cdot u \times v = B_x \, e_x \cdot u \times v + B_y \, e_y \cdot u \times v + B_z \, e_z \cdot u \times v$$

$$= B_x \det\begin{bmatrix} 1 & u_x & v_x \\ 0 & u_y & v_y \\ 0 & u_z & v_z \end{bmatrix} + B_y \det\begin{bmatrix} 0 & u_x & v_x \\ 1 & u_y & v_y \\ 0 & u_z & v_z \end{bmatrix} + B_z \det\begin{bmatrix} 0 & u_x & v_x \\ 0 & u_y & v_y \\ 1 & u_z & v_z \end{bmatrix}$$

$$= \underbrace{B_x \det\begin{bmatrix} - & - \\ u_y & v_y \\ u_z & v_z \end{bmatrix} - B_y \det\begin{bmatrix} u_x & v_x \\ - & - \\ u_z & v_z \end{bmatrix} + B_z \det\begin{bmatrix} u_x & v_x \\ u_y & v_y \\ - & - \end{bmatrix}}_{\text{the three maximal minors for 2 vectors in } \mathbb{R}^3}$$

Definition: A differential k-form on $\mathbb{R}^n$ is a k-form field. I.e., if γ is a differential k-form and $x \in \mathbb{R}^n$ then γ_x is a k-form.

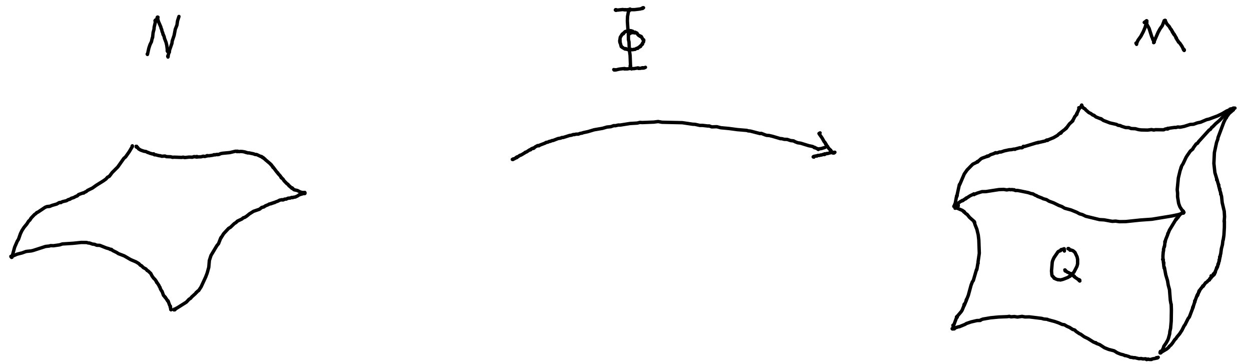
Example

Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be a magnetic field.

- The <u>magnetic</u> <u>circulation</u> $B \cdot d\ell$ is a differential 1-form,
  $(B \cdot d\ell)_x (u) = B(x) \cdot u$.
- The <u>magnetic</u> <u>flux</u> $B \cdot dS$ is a differential 2-form,
  $(B \cdot dS)_x (u, v) = B(x) \cdot u \times v$.
- <u>any</u> <u>k-form</u> on $\mathbb{R}^3$ is also a (constant) differential k-form.
  e.g. the <u>standard</u> <u>volume</u> <u>form</u> d^3x is a diff. 3-form,
  $(d^3x)_x (u, v, w) = u \cdot v \times w$

(II) Pullbacks

Idea behind pullbacks



We are given

- a map Φ between spaces N and M
- Q a known object on M

We want

- an object of same type on N

When we can do it

we call it

The pullback of Q along Φ

$\Phi^* Q$

(lives on N)

Pullback formulas

---

vectors $N = M = \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ linear + invertible, $Q = v \in \mathbb{R}^n$

$$\Phi^* v = \Phi^{-1} v .$$

---

k-forms $N = M = \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ linear, $Q = \gamma \in \Lambda^k(\mathbb{R}^n)$

$$(\Phi^* \gamma)(v_1, \dots, v_k) = \gamma(\Phi v_1, \dots, \Phi v_k)$$

---

vector fields $N = M = \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism, $Q = B$ a vector field

$$(\Phi^* B)(x) = (D\Phi(x))^{-1} B(\Phi(x))$$

---

differential k-forms $N = M = \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ smooth, $Q = \beta$ a differential k-form

$$(\Phi^* \beta)_x (v_1, \dots, v_k) = \beta_{\Phi(x)}(D\Phi(x) v_1, \dots, D\Phi(x) v_k)$$

Example

- Given 3x3 invertible matrix $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  let's compute pullback of the 2-form

  $$\gamma : (u,v) \longmapsto B \cdot u \times v$$

- $$(L^*\gamma)(u,v) = \gamma(Lu, Lv) = B \cdot Lu \times Lv = \det\begin{bmatrix} | & | & | \\ B & Lu & Lv \\ | & | & | \end{bmatrix}$$

$$= \det\left(L L^{-1} \begin{bmatrix} | & | & | \\ B & Lu & Lv \\ | & | & | \end{bmatrix} \right)$$

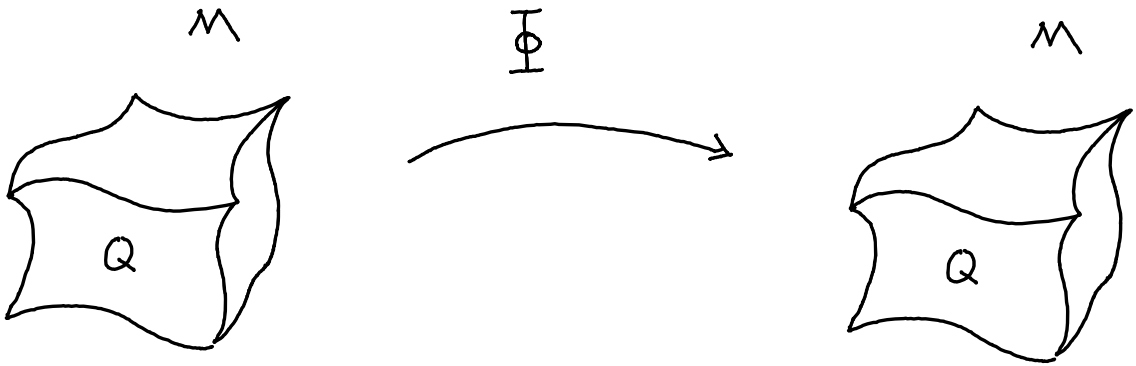
$$= \det(L) \det\begin{bmatrix} | & | & | \\ L^{-1}B & u & v \\ | & | & | \end{bmatrix}$$

$$= \det(L) \left(L^{-1}B\right) \cdot u \times v$$

$$\Rightarrow \boxed{(L^*\gamma)(u,v) = B' \cdot u \times v \quad , \quad B' = \det(L) L^{-1} B}$$

(III) Symmetry

A general notion of symmetry



M : a space

Q : an object on M

An invertible map $\Phi : M \longrightarrow M$ is a <u>symmetry</u> of Q if

$$\Phi^* Q = Q$$

Example

- For $M = \mathbb{R}^3$, $Q = B \in \mathbb{R}^3$, let's find all linear symmetries $L : \mathbb{R}^3 \to \mathbb{R}^3$ of B (a vector)

- Recall $\boxed{L^* B = L^{-1} B}$

$\Rightarrow$ if L is a symmetry of then

$$\{ B = L^{-1} B \} \quad (S)$$

- choose basis for $\mathbb{R}^3$ (e_1, e_2, e_3) s.t. $e_1 = B$. In this basis, condition (S) becomes

$$\begin{bmatrix} L_{11} \\ L_{21} \\ L_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \iff L = \left[\begin{array}{c|cc} 1 & E & F \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & & \end{array}\right] \iff \left\{ L = \left[\begin{array}{c|cc} 1 & E & F \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & & \end{array}\right] \quad \begin{array}{l} E, F \in \mathbb{R} \\ \det(C) \neq 0 \end{array} \right\}$$

$$\Downarrow$$

$$\det(L) = \det(C) \neq 0$$

Example

- For $M=\mathbb{R}^3$, $Q=\gamma\in\Lambda^2(\mathbb{R}^3)$, $\gamma(u,v)=B\cdot u\times v$, let's find all linear symmetries $L:\mathbb{R}^3\to\mathbb{R}^3$ of γ.

- Recall $\boxed{(L^*\gamma)(u,v)=B'\cdot u\times v \quad , \quad B'=\det(L)\,L^{-1}B}$

$\Rightarrow$ if L is a symmetry of γ then $B'=B$, or

$$\left\{\det(L)\,L^{-1}B=B\right\} \qquad (S)$$

- choose basis for $\mathbb{R}^3$ (e_1,e_2,e_3) s.t. $e_1=B$. In this basis, condition (S) becomes

$$\begin{bmatrix} L_{11}\\ L_{21}\\ L_{31}\end{bmatrix}=\det(L)\begin{bmatrix}1\\0\\0\end{bmatrix} \iff L=\left[\begin{array}{c|cc} \det(L) & E & F \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{C} \end{array}\right] \iff \left\{ L=\left[\begin{array}{c|cc} D & E & F \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{C} \end{array}\right] \quad \begin{array}{c} D\neq 0 \\ E,F\in\mathbb{R} \\ \det(C)=1 \end{array} \right\}$$

$$\Downarrow$$

$$\det(c)=1 \Leftarrow \det(L)=\det(L)\det(C)$$

Attention!!

Although B and $(u,v) \mapsto B \cdot u \times v$ encode the same vector B, their linear symmetries are <u>not</u> the same.

$\gamma : (u,v) \mapsto B \cdot u \times v$

$$L = \left[\begin{array}{c|cc} D & E & F \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & & \end{array}\right] \qquad \begin{array}{c} D \neq 0 \\ E, F \in \mathbb{R} \\ \det(C) = 1 \end{array}$$

B

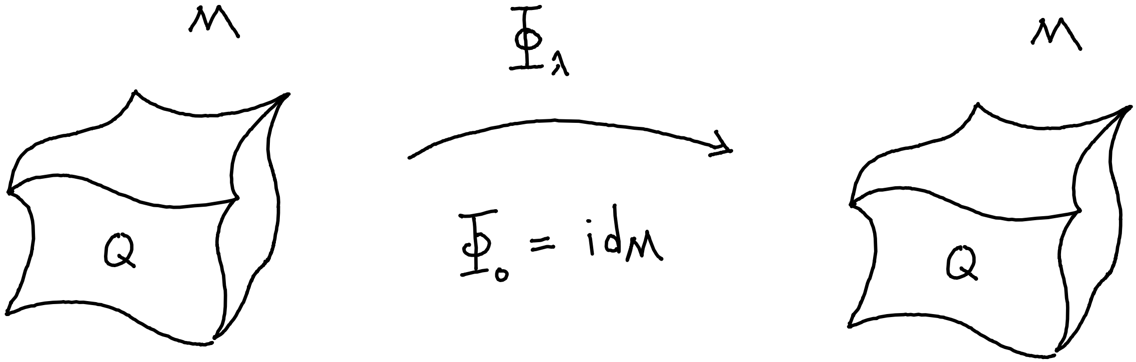
$$L = \left[\begin{array}{c|cc} 1 & E & F \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & & \end{array}\right] \qquad \begin{array}{c} E, F \in \mathbb{R} \\ \det(C) \neq 0 \end{array}$$

L sym. of γ $\not\Rightarrow$ L sym. of B

L sym. of B $\not\Rightarrow$ L sym. of γ

Infinitesimal symmetry

$\lambda \in \mathbb{R}$



- Suppose $\Phi_\lambda : M \to M$ is a smooth 1-parameter family of diffeos
- $u = \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_\lambda$ defines a vector field on M
- $\mathcal{L}_u = \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_\lambda^*$ defines a linear operator on Q-space called <u>Lie</u> <u>derivative</u> <u>along</u> <u>u</u>
- $\Phi_\lambda^* Q = Q \Rightarrow \mathcal{L}_u Q = 0$
- $\mathcal{L}_u Q = 0 \Rightarrow \exp(\lambda u)$ is 1-parameter family of sym's

Definition: A vector field u on a space M is an *infinitesimal symmetry* of an object Q on M if
$$\mathcal{L}_u Q = 0$$

$\boxed{\text{Example}}$

- Let's find linear infinitesimal symmetries of $\gamma \in \Lambda^2(\mathbb{R}^3)$ given by $\gamma(u,v) = B \cdot u \times v$.
- $L(\lambda)$ is 1-parameter family of 3×3 matrices. $U = \frac{d}{d\lambda}\Big|_0 L(\lambda)$
- $(\mathcal{L}_u \gamma)(w,v) = \left(\frac{d}{d\lambda}\Big|_0 L(\lambda)^*\gamma\right)(w,v) = \frac{d}{d\lambda}\Big|_0 \left[\det(L(\lambda))L^{-1}(\lambda)B\right] \cdot w \times v$

$$= \left[Tr(U)B - UB\right] \cdot w \times v$$

$\Longrightarrow$ U is int. sym. iff $\boxed{UB = Tr(U)B}$

$\Longleftrightarrow$ in basis (e_1, e_2, e_3) w/ $B = e_1$

$$\boxed{U = \left[\begin{array}{c|cc} d & a & b \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{\mathcal{C}} \end{array}\right] \qquad \begin{array}{l} d,a,b \in \mathbb{R} \\ Tr(\mathcal{C}) = 0 \end{array}}$$

Infinitesimal symmetries of magnetic fields

- Let B be a magnetic field on $\mathbb{R}^3$. $(\nabla \cdot B = 0)$

- The condition for u to be an inf. sym. of

  * B (vector field) is $\boxed{u \cdot \nabla B - B \cdot \nabla u = 0}$
  * $B \cdot d\ell$ (diff. 1-form) is $\boxed{(\nabla \times B) \times u + \nabla (u \cdot B) = 0}$
  * $B \cdot dS$ (diff. 2-form) is $\boxed{\nabla \times (u \times B) = 0}$
  * $|B|^2$ (diff. 0-form) is $\boxed{u \cdot \nabla |B| = 0}$

- An (infinitesimal) Quasisymmetry is a vector field u that is simultaneously an inf. sym. for

  $B \cdot d\ell$, $B \cdot dS$, and $|B|^2$

An important lesson

- Whether you view a magnetic field as
    - a vector field B
    - a diff. 1-form $B \cdot d\ell$
    - a diff. 2-form $B \cdot dS$

  is a matter of bookkeeping.

- However, the notions of symmetry for
    - B
    - $B \cdot d\ell$
    - $B \cdot dS$

  are inherenty different. This is sufficient motivation to understand differential forms as a plasma physicist.