Symmetries of magnetic fields:

When symmetry breaks

Outline

(I) Field line variational principle

(II) Island formation

(III) Chaos (Nathan)

Outline

(I) Field line variational principle

Hamilton's principle applies to field lines

- A vector potential for B

- $c : [\lambda_1, \lambda_2] \longrightarrow \mathbb{R}^3$ parameterized curve

- Field line "action":

$$S(c) = \int_C A \cdot dl = \int_{\lambda_1}^{\lambda_2} A(c(\lambda)) \cdot \frac{dc}{d\lambda} \, d\lambda$$

- First variation formula:

$$\delta S(c) = A \cdot \delta c \Big|_{\lambda_1}^{\lambda_2} + \int_{\lambda_1}^{\lambda_2} B \cdot \delta c \times \frac{dc}{d\lambda} \, d\lambda$$

- Euler-Lagrange equation:

$$\boxed{B(c(\lambda)) \times \frac{dc}{d\lambda} = 0}$$

Let's use Hamilton's principle to dissect breakdow of symmetry

Symmetric point of departure

- coordinates $\quad (\theta, \xi, \psi) \in S^1 \times S^1 \times \mathbb{R}$
- $A = \psi \nabla\theta - \psi_p(\psi) \nabla\xi$
- 2-parameter family of transformations

$$\Phi_{\Delta\theta, \Delta\xi}(\theta, \xi, \psi) = (\theta + \Delta\theta, \xi + \Delta\xi, \psi)$$

- transformation of parameterized curve:

$$c : [\lambda_1, \lambda_2] \longrightarrow S^1 \times S^1 \times \mathbb{R} \quad : \quad \lambda \longmapsto (\theta(\lambda), \xi(\lambda), \psi(\lambda))$$

$$\Phi_{\Delta\theta, \Delta\xi}(c) : [\lambda_1, \lambda_2] \longrightarrow S^1 \times S^1 \times \mathbb{R} \quad : \lambda \longmapsto (\theta(\lambda) + \Delta\theta, \xi(\lambda) + \Delta\xi, \psi(\lambda))$$

Claim: $\Phi_{\Delta\theta, \Delta\xi}$ is a 2-parameter family of symmetries of field line action S

Two field-line conservation laws

$$S(\Phi_{\Delta\theta,\Delta\zeta}(c)) = S(c) \quad , \quad B \times \frac{dc}{d\lambda} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial \Delta\theta}\bigg|_0 S(\Phi_{\Delta\theta,\Delta\zeta}(c)) = 0 \quad , \quad \frac{\partial}{\partial \Delta\zeta}\bigg|_0 S(\Phi_{\Delta\theta,\Delta\zeta}(c)) = 0$$

$$\Rightarrow \quad \delta S(c) = 0 \quad , \quad \delta S(c) = 0$$

$$\delta c(\lambda) = (1,0,0) \quad , \quad \delta c(\lambda) = (0,1,0)$$

$$\Rightarrow \quad 0 = A(c(\lambda)) \cdot \delta c(\lambda)\Big|_{\lambda_1}^{\lambda_2} \quad , \quad 0 = A(c(\lambda)) \cdot \delta c(\lambda)\Big|_{\lambda_1}^{\lambda_2}$$

$$= (\psi, -\psi_p, 0) \cdot (1,0,0)\Big|_{\lambda_1}^{\lambda_2} \quad , \quad = (\psi, -\psi_p(\psi), 0) \cdot (0,1,0)\Big|_{\lambda_1}^{\lambda_2}$$

$$= \psi(\lambda_2) - \psi(\lambda_1) \quad , \quad = -\psi_p(\psi(\lambda_2)) + \psi_p(\psi(\lambda_1))$$

Outline

(II) Island formation

A single mode perturbation

$$A = \psi \nabla\theta - \psi_p(\psi)\nabla\zeta + \epsilon\, a_\theta(\psi)\sin(m\theta + n\zeta + \varphi_\theta(\psi))\nabla\theta$$
$$+ \epsilon\, a_\zeta(\psi)\sin(m\theta + n\zeta + \varphi_\zeta(\psi))\nabla\zeta$$
$$+ \epsilon\, a_\psi(\psi)\sin(m\theta + n\zeta + \varphi_\psi(\psi))\nabla\psi$$

- The field line action is not invariant under $\Phi_{\Delta\theta,\Delta\zeta}$ for general $\Delta\theta, \Delta\zeta$

$$S(\Phi_{\Delta\theta,\Delta\zeta}(c)) \neq S(c)$$

symmetry is broken by the perturbation!

Why?

- Let $\Phi_{\Delta\theta,\Delta\zeta}(c)(\lambda) = (\theta'(\lambda), \zeta'(\lambda), \psi'(\lambda))$.
- We have

$$\theta'(\lambda) = \theta(\lambda) + \Delta\theta$$
$$\zeta'(\lambda) = \zeta(\lambda) + \Delta\zeta$$
$$\psi'(\lambda) = \psi(\lambda)$$

- The Key issue:

$$\sin(m\theta'(\lambda) + n\zeta'(\lambda) + \varphi(\psi'(\lambda)))$$
$$= \sin(m\theta(\lambda) + n\zeta(\lambda) + \varphi(\psi(\lambda)) + m\Delta\theta + n\Delta\zeta)$$

Symmetry is not completely broken by one mode

- Define $\Phi_\xi = \Phi_{n\xi, -m\xi}$ $\leftarrow$ a 1-parameter subset of original 2-parameter collection of symmetries

- Note $\Phi_\xi(c)(\lambda) = (\theta'(\lambda), \xi'(\lambda), \psi'(\lambda))$, with

$$\begin{aligned} \theta'(\lambda) &= \theta(\lambda) + n\xi \\ \xi'(\lambda) &= \xi(\lambda) - m\xi \\ \psi'(\lambda) &= \psi(\lambda) \end{aligned} \quad \Rightarrow \quad \begin{aligned} m\Delta\theta + n\Delta\xi &= mn\xi - nm\xi \\ &= 0 \end{aligned}$$

$$\Rightarrow \sin(m\theta'(\lambda) + n\xi'(\lambda) + \varphi(\psi'(\lambda))) = \sin(m\theta(\lambda) + n\xi(\lambda) + \varphi(\psi(\lambda)))$$

$$\Rightarrow \boxed{S(\Phi_\xi(c)) = S(c)}$$

Noether's theorem gives one conservation law

$$\partial_\xi\Big|_0 S(\Phi_\xi(c)) = 0 \quad , \quad B \times \frac{dc}{d\lambda} = 0$$

$$\Rightarrow \quad \delta S(c) = 0 \quad , \quad \delta c(\lambda) = (\delta\theta(\lambda), \delta\zeta(\lambda), \delta\psi(\lambda)) = (n, -m, 0)$$

$$\Rightarrow \quad 0 = A(c(\lambda)) \cdot \delta c(\lambda)\Big|_{\lambda_1}^{\lambda_2} \quad \text{(by first variation formula)}$$

$$= A(c(\lambda)) \cdot (n, -m, 0)\Big|_{\lambda_1}^{\lambda_2}$$

$$= n A_\theta - m A_\zeta \Big|_{\lambda_1}^{\lambda_2}$$

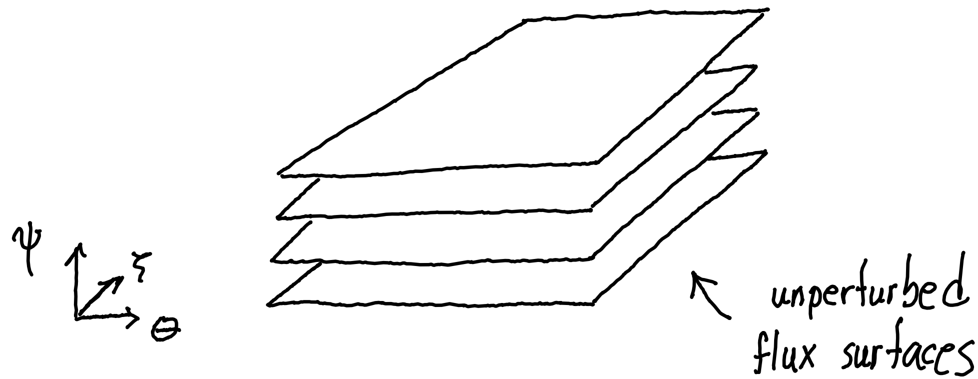
$$= \left\{ n\psi - m\psi_p(\psi) + \epsilon\left[n\, a_\theta(\psi) \sin(m\theta + n\zeta + \varphi_\theta(\psi)) - m\, a_\zeta(\psi) \sin(m\theta + n\zeta + \varphi_\zeta(\psi)) \right] \right\}\Big|_{\lambda_1}^{\lambda_2}$$

$$= \left\{ n\psi - m\psi_p(\psi) + \epsilon\, a_\zeta(\psi) \sin(m\theta + n\zeta + \varphi_\zeta(\psi)) \right\}\Big|_{\lambda_1}^{\lambda_2}$$

$$= \psi_\xi(\theta, \zeta, \psi)\Big|_{\lambda_1}^{\lambda_2}$$

What do level sets of ψ_ξ look like?

Before perturbation:



After perturbation:

?

Let's try to use perturbation theory to understand

- $\psi_\xi = n\psi - m\psi_p(\psi) + \epsilon\, a_\xi(\psi)\sin(m\theta + n\zeta + \varphi_\xi(\psi))$

- solve $\psi_\xi(\theta, \zeta, \psi) = \Gamma$ for ψ

  $\uparrow$ label for perturbed flux surface

- technique: $\rightarrow$ let $\psi = \psi^*(\theta, \zeta)$

  $\rightarrow$ expand $\psi^* = \psi^*_0 + \epsilon\psi^*_1 + \epsilon^2\psi^*_2 + \dots$

  $\rightarrow$ solve for ψ^*_k sequentially

Let's try to use perturbation theory to understand

- $$\psi_\xi = n\psi - m\psi_p(\psi) + \epsilon\, a_\xi(\psi)\sin(m\theta + n\xi + \varphi_\xi(\psi))$$

leading order : $$n\psi_0^*(\theta,\xi) - m\psi_p(\psi_0^*(\theta,\xi)) = \Gamma \Rightarrow \boxed{\psi_0^* = \text{const.}}$$

first order : $$(n - m\psi_p'(\psi_0^*))\,\psi_1^* + a_\xi(\psi_0^*)\sin(m\theta + n\xi + \varphi_\xi(\psi_0^*)) = 0$$

$\vdots$

n'th order : $$(n - m\psi_p'(\psi_0^*))\,\psi_n^* = \text{L.O.T.}$$

$\Rightarrow$ expansion only works if

$$n - m\psi_p'(\psi_0^*) = n - m\,\iota(\psi_0^*) \neq 0$$

General picture for single mode perturbation

→ flux surfaces $\psi = \psi_0^* =$ const. w/

$$n - m\,\iota(\psi_0^*) \neq 0$$

are gently deformed

→ flux surfaces $\psi = \psi_0^* =$ const. w/

$$n - m\,\iota(\psi_0^*) = 0$$

do something more complicated . . .

an island forms!