Magnetic coordinates

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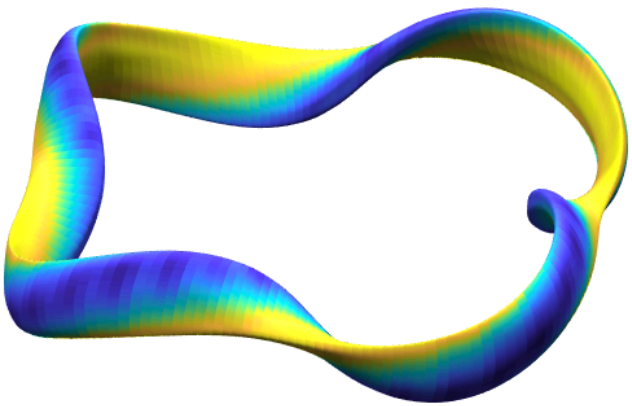
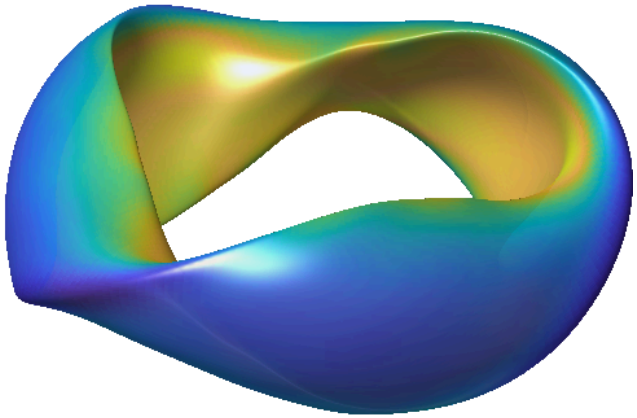
An Introduction to Stellarators (arXiv :1908.05360)
From magnetic fields to symmetries and optimization

- Self-contained and introductory
- Scope : challenges in stellarator design
- Goal : Stimulate cross-disciplinary collaboration





Today

- Convenient coordinate systems
- Toward the magnetic differential equation

Context

Stellarator design

- Equillibrium magnetic field with good (confinement) properties
- Magnetic coils to generate this magnetic field
- Two-stage vs one-stage approach

Fundamental

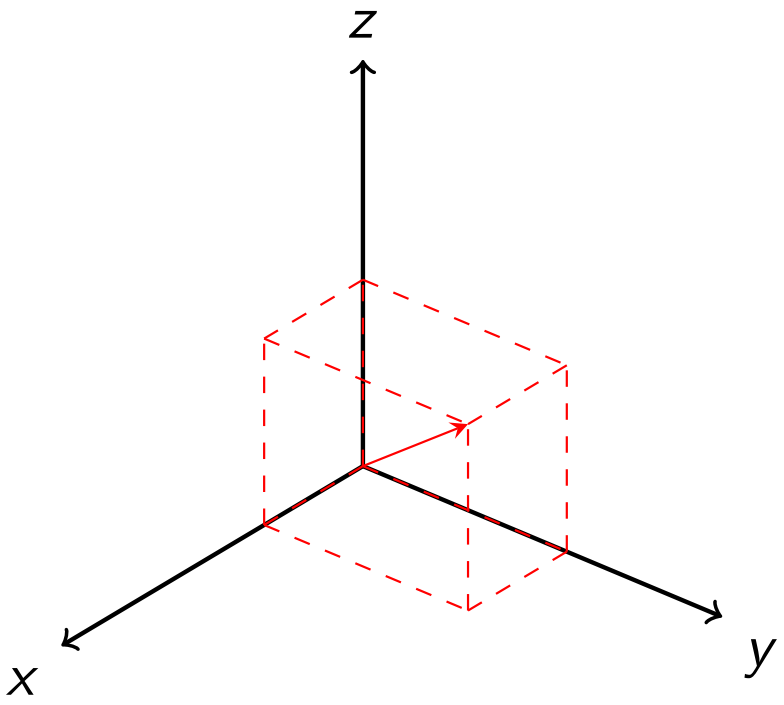
- Problem formulation
  - Coordinate systems
- Assumptions on magnetic field
  - Desirable properties of the unknown magnetic field

Non-orthogonal coordinate systems

Examples of coordinate systems

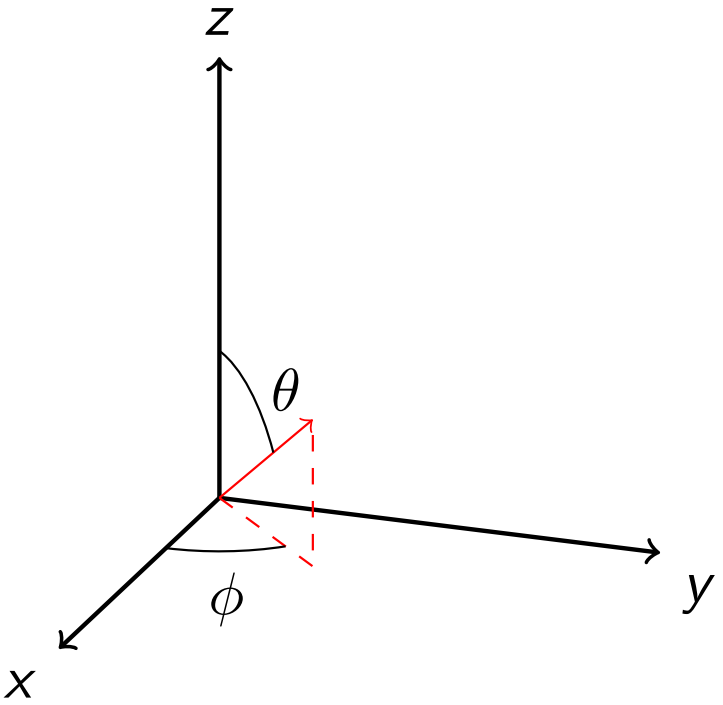
Cartesian coordinates

- (x, y, z)



Spherical coordinates

- (r, θ, ϕ)



$\Rightarrow$ Choose coordinate system adapted to problem of interest

Orthogonal coordinate systems

Definition

- Coordinate surfaces
  - Surfaces on which one coordinate is a constant

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3, x_i \text{ constant } \right\} \text{ for i } \in \{1, 2, 3\}$$

- Orthogonal coordinates
  - At every point in space
  - Coordinate surfaces **orthogonal to each other** (their normals)

Cartesian coordinates

- $(x, y, z) \in \mathbb{R}^3$
- Coordinates surfaces
  - Planar for the three coordinates

Spherical coordinates

- $(r, \theta, \phi) \in \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi)$
- Coordinates surfaces
  - Spherical (r), Conical (θ), half planar (ϕ)

A non-orthogonal coordinate system

Cylindrical coordinates

- (r, θ, z)
  - $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
  - $\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = atan2(y, x) \end{cases}$
- Coordinates surfaces
  - Cylindrical (r), half planar (θ), planar (z)

Stretched cyl. coordinates

- (ρ, ζ, z)
  - $\begin{cases} x = 2\rho\cos\zeta \\ y = \rho\sin\zeta \end{cases}$
  - $\begin{cases} \rho = \sqrt{\left(\frac{x}{2}\right)^2 + y^2} \\ \zeta = atan2\left(y, \frac{x}{2}\right) \end{cases}$
- Coordinates surfaces
  - Elliptic-cylindrical (ρ), curved (ζ), planar (z)

Coordinate systems and associate bases

Given a coordinate system (x_1, x_2, x_3)

- Definition of **local** bases at any point $\boldsymbol{r}(x_1, x_2, x_3) \in \mathbb{R}^3$

Contravariant basis

- basis of the derivatives of the position vector
- $\left(\dfrac{\partial \boldsymbol{r}}{\partial x_1}, \dfrac{\partial \boldsymbol{r}}{\partial x_2}, \dfrac{\partial \boldsymbol{r}}{\partial x_3}\right)$

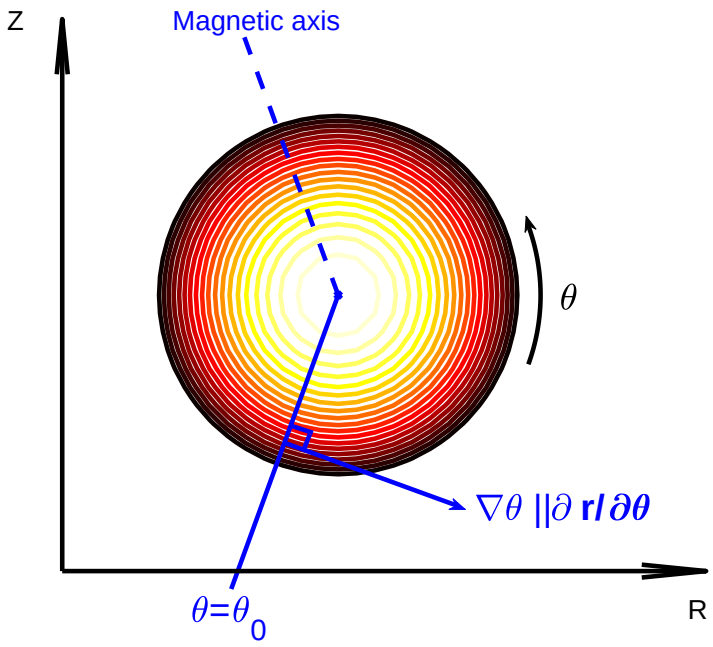
Covariant basis

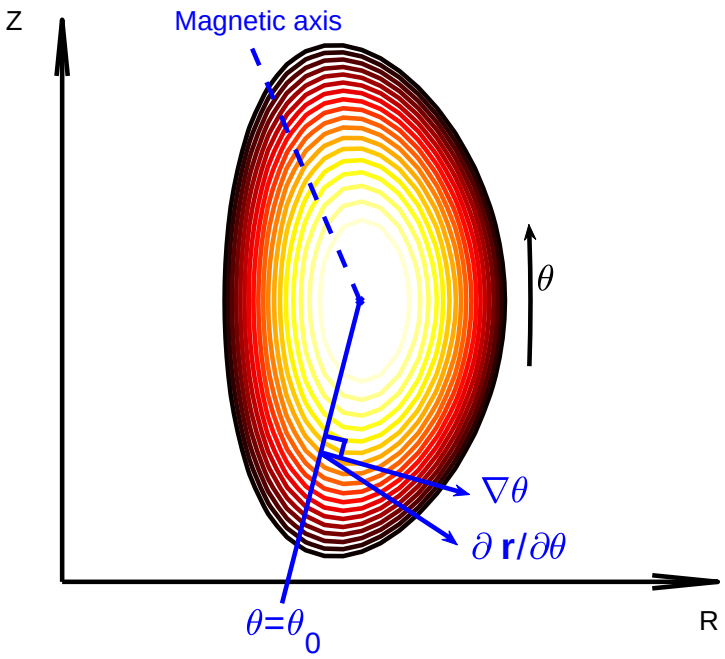
- basis of the gradients of the coordinates
- $(\nabla x_1, \nabla x_2, \nabla x_3)$

Duality property

- $\nabla x_i \cdot \dfrac{\partial \boldsymbol{r}}{\partial x_j} = \delta_{ij}$
- Orthogonality $(i \neq j)$
- Normalization $(i = j)$

Examples of orthogonal and non-orthogonal (flux) coordinate systems in a poloidal plane





Vector fields and generic coordinate systems (x_1, x_2, x_3)

Vector field components

- Covariant form $\boldsymbol{A} = \sum_{i=1}^{3} A_i \nabla x_i$ with $A_i = \boldsymbol{A} \cdot \partial \boldsymbol{r} / \partial x_i$
- Contravariant form $\boldsymbol{A} = \sum_{i=1}^{3} A^i \frac{\partial \boldsymbol{r}}{\partial x_i}$ with $A^i = \boldsymbol{A} \cdot \nabla x_i$

Derivatives

- Divergence $\nabla \cdot \boldsymbol{A} = \sum_{i=1}^{3} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} A^i \right)$
- Curl $\nabla \times \boldsymbol{A} = \sum_{k=1}^{3} \frac{1}{\sqrt{g}} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \frac{\partial \boldsymbol{r}}{\partial x_k}$
- Gradient $\nabla q = \sum_{i=1}^{3} \frac{\partial q}{\partial x_i} \nabla x_i$

Non-orthogonal coordinate systems (x_1, x_2, x_3)

Other useful formulas

- Jacobian
$$\sqrt{g} = \left(\frac{\partial \boldsymbol{r}}{\partial x_1} \times \frac{\partial \boldsymbol{r}}{\partial x_2}\right) \cdot \frac{\partial \boldsymbol{r}}{\partial x_3} = ((\nabla x_1 \times \nabla x_2) \cdot \nabla x_3)^{-1}$$
- Relation between basis vectors
$$\frac{\partial \boldsymbol{r}}{\partial x_k} = \sqrt{g}\,(\nabla x_i \times \nabla x_j) \text{ and } \nabla x_k = \sqrt{g}^{-1}\left(\frac{\partial \boldsymbol{r}}{\partial x_i} \times \frac{\partial \boldsymbol{r}}{\partial x_j}\right)$$
- Differential volume
$$d^3x = |\sqrt{g}|dx_1 dx_2 dx_3$$
- Differential surface
$$d^2x = |\sqrt{g}||\nabla x_k|dx_i dx_j \text{ (const. } x_k)$$

Closed nested toroidal flux surfaces

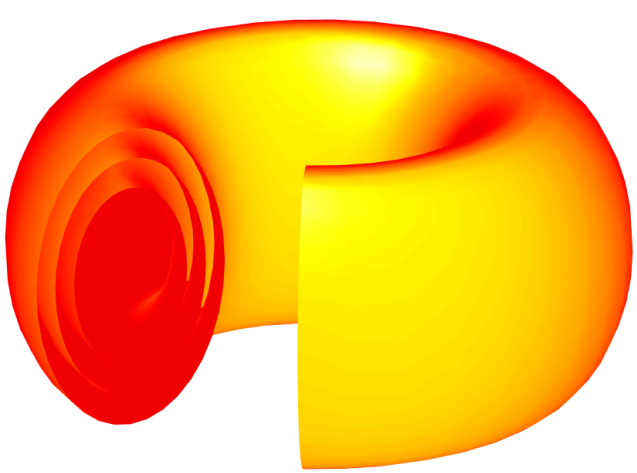
Fundamental simplifying assumption

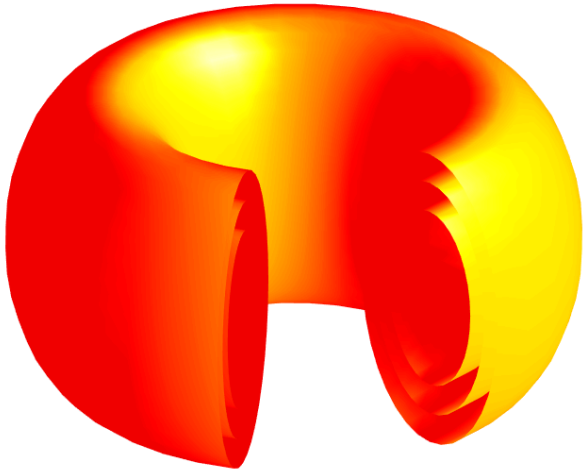
Flux surfaces and flux labels

Motivation

- Avoid field lines connecting the plasma core to the wall
- Toroidal magnetic surface
  - smooth surface, with normal $\hat{\boldsymbol{n}}$
  - $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0$ at every point on the surface
  - Field lines wrap around closed toroidal flux surfaces
- Ideally set of continuously nested closed toroidal surfaces
  - Nested around a single closed field line, the *magnetic axis*

Assuming existence of nested toroidal flux surfaces





Flux surfaces and flux labels

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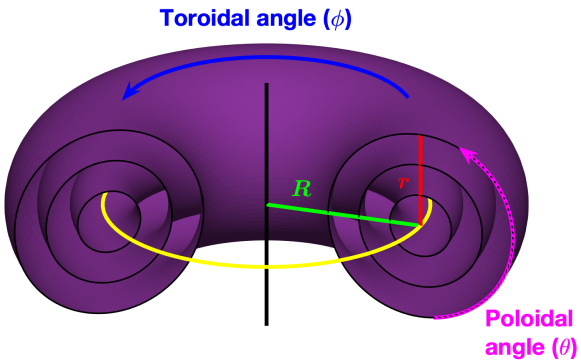
Assuming existence of nested toroidal flux surfaces

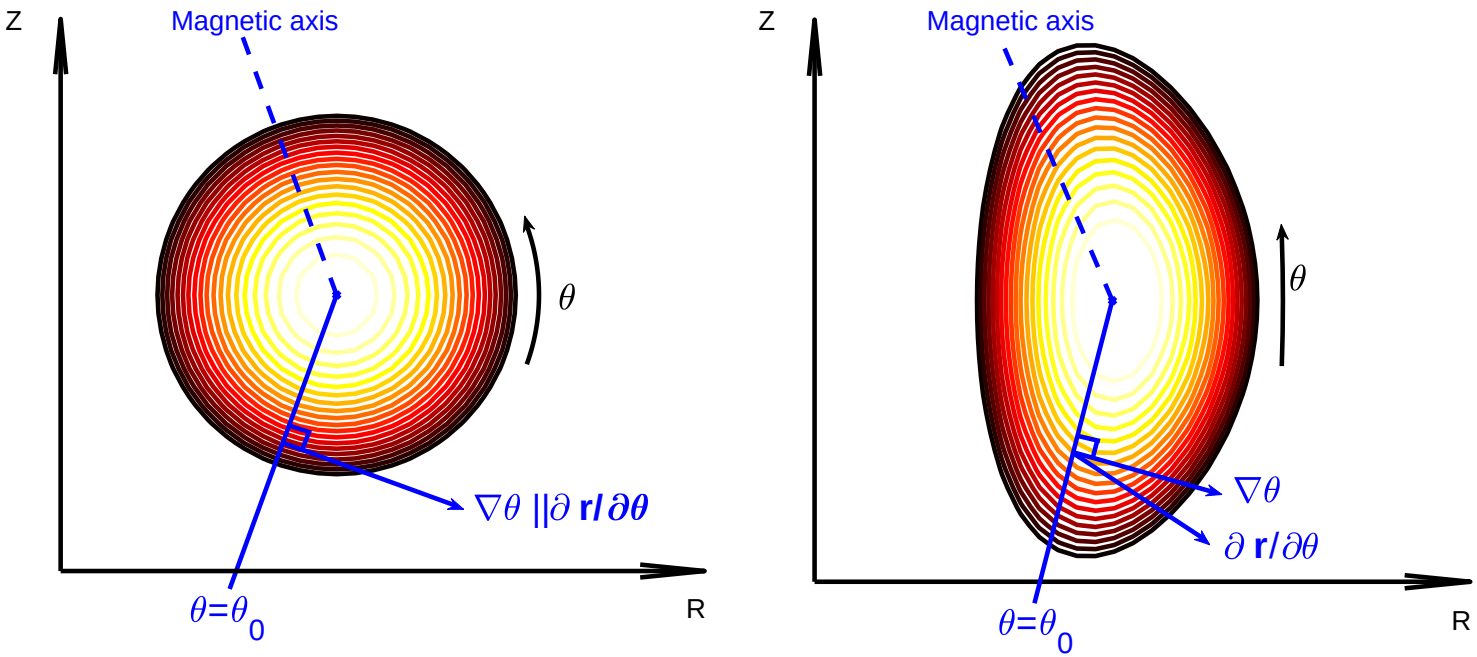
- Flux function
  - Quantity constant on a flux surface
- Flux label
  - Smooth one-to-one real-valued fct defined on set of surfaces
  - Defined on the set of surfaces, monotonic wrt distance to axis
  - ✓ Each surface is uniquely labeled

Flux coordinates

Flux coordinates (r, θ, ϕ), defined from the magnetic field

- Assuming existence of nested surfaces
- A flux surface label $r \in \mathbb{R}^+$ (0 on axis)
  A poloidal angles $\theta \in [0, 2\pi)$
  A toroidal angle $\phi \in [0, 2\pi)$
- Local co and contravariant bases



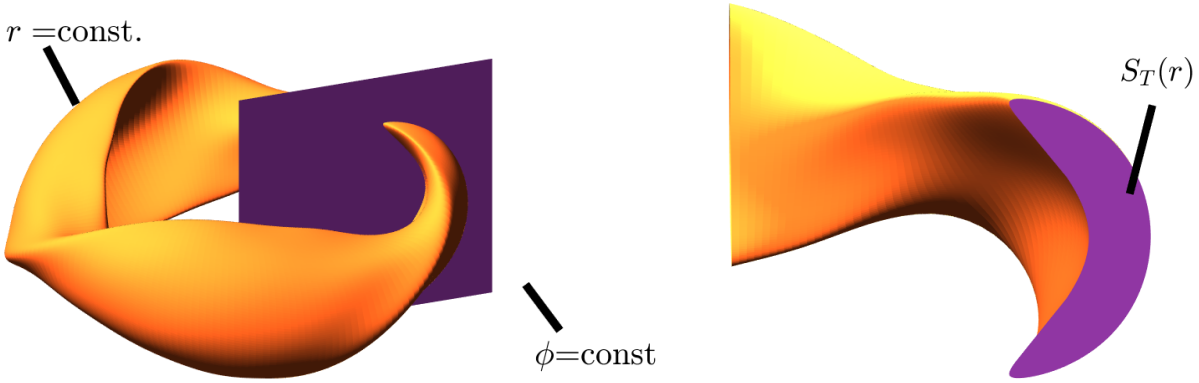


Curves of constant flux in a poloidal plane

Standard flux label ψ

Toroidal flux across a poloidal cut

- $\Psi_T(r) = \int_0^r \int_0^{2\pi} \left[\frac{\boldsymbol{B} \cdot \nabla\phi}{\nabla r \cdot \nabla\theta \times \nabla\phi} \right] (\mathbf{r}(\tilde{r}, \theta, \phi_0))\, d\theta d\tilde{r}$



- $S_T(r) = \left\{ \mathbf{r}(\tilde{r}, \theta, \phi_0), \tilde{r} \in [0, r], \theta \in [0, 2\pi) \right\}$
- Flux surface $r \Rightarrow$ indep. of $\phi_0 \Rightarrow \Psi_T$ is a flux label

Define $\psi = \Psi_T / 2\pi$ $\Rightarrow (\psi, \theta, \phi)$

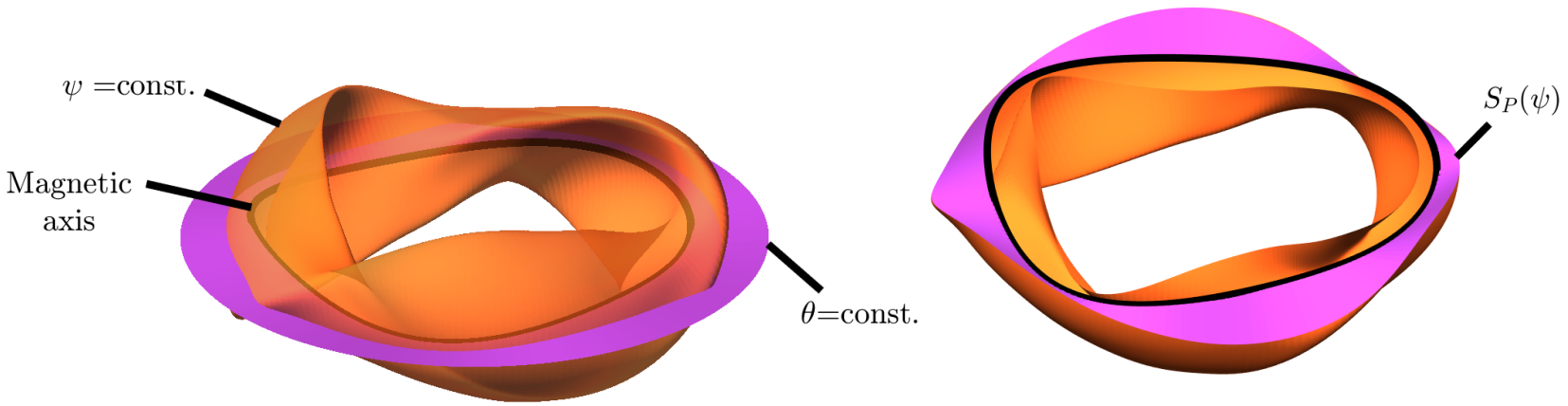
Magnetic field property

- Normal to flux surface $\hat{\boldsymbol{n}}$ aligned with $\partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r} = g^{-1/2} \nabla\psi$ so $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0 \Rightarrow \boldsymbol{B} \cdot \nabla\psi = 0$

Poloidal flux and rotational transform

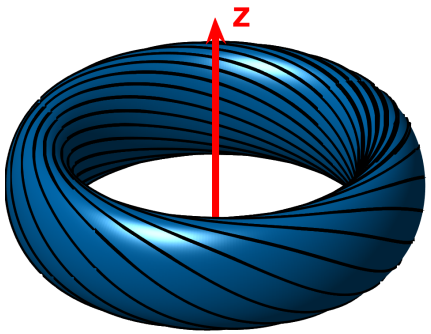
Poloidal flux across a toroidal cut

- $\Psi_P(\psi) = \int_0^{\psi} \int_0^{2\pi} \left[\frac{\boldsymbol{B} \cdot \nabla\theta}{\nabla\psi \cdot \nabla\theta \times \nabla\phi} \right] \left(\mathbf{r}(\widetilde{\psi}, \theta_0, \phi) \right) d\phi d\widetilde{\psi}$



Definition

- $\iota(\psi) = \frac{d\Psi_P(\psi)/d\psi}{d\Psi_T(\psi)/d\psi}$
- Rotational transform



Magnetic coordinates

Straight field lines coordinates

Magnetic coordinates Assuming $\nabla \cdot \boldsymbol{B} = 0$

- Contravariant form of the field

$$\begin{aligned}
\boldsymbol{B} &= B^{\psi}\partial_{\psi}\boldsymbol{r} + B^{\theta}\partial_{\theta}\boldsymbol{r} + B^{\phi}\partial_{\phi}\boldsymbol{r} \\
&= B^{\theta}\partial_{\theta}\boldsymbol{r} + B^{\phi}\partial_{\phi}\boldsymbol{r} && \text{since } \boldsymbol{B}\cdot\nabla\psi = 0 \\
&= B^{\theta}\sqrt{g}\nabla\phi \times \nabla\psi + B^{\phi}\sqrt{g}\nabla\psi \times \nabla\theta \\
&= \nabla\psi \times \left(-B^{\theta}\sqrt{g}\nabla\phi + B^{\phi}\sqrt{g}\nabla\theta\right)
\end{aligned}$$

- Divergence of the field

$$\begin{aligned}
\nabla \cdot \boldsymbol{B} = 0 \quad &\Rightarrow \partial_{\theta}\left(B^{\theta}\sqrt{g}\right) + \partial_{\phi}\left(B^{\phi}\sqrt{g}\right) = 0 \\
&\Rightarrow \exists\ v \text{ s.t. } \begin{cases} \partial_{\phi} v = -B^{\theta}\sqrt{g} \\ \partial_{\theta} v = B^{\phi}\sqrt{g} \end{cases} \quad \text{so } \boldsymbol{B} = \nabla\psi \times \nabla v
\end{aligned}$$

- **Periodicity**

$\boldsymbol{B}$ single-valued $\Rightarrow \nabla v$ single-valued so periodic wrt (θ, ϕ)
$\Rightarrow \exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.

$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$

Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Toroidal flux
$$\begin{aligned}
\Psi_T(\psi) &= \int_0^{\psi}\int_0^{2\pi} \left[\sqrt{g}\,\underbrace{\boldsymbol{B}\cdot\nabla\phi}_{B^{\phi}}\right]\left(\mathbf{r}(\widetilde{\psi}, \theta, \phi_0)\right)\, d\theta d\widetilde{\psi} \\
&= \int_0^{\psi}\int_0^{2\pi} \partial_\theta v\left(\mathbf{r}(\widetilde{\psi}, \theta, \phi_0)\right)\, d\theta d\widetilde{\psi} \\
&= \int_0^{\psi}\int_0^{2\pi} j(\widetilde{\psi})\, d\theta d\widetilde{\psi} \text{ as } \lambda \text{ is periodic} \\
&= 2\pi\int_0^{\psi} j(\widetilde{\psi})\, d\widetilde{\psi}
\end{aligned}$$

Hence
$$\Psi_T(\psi) = 2\pi\psi \Rightarrow j(\psi) = 1$$

Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Toroidal flux $\Rightarrow j(\psi) = 1$

Simplified contravariant form

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$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Poloidal flux

$$\begin{aligned}
\Psi_P(\psi) &= \int_0^{\psi}\int_0^{2\pi} \left[\sqrt{g}\,\underbrace{\boldsymbol{B}\cdot\nabla\theta}_{B^\theta}\right]\left(\mathbf{r}(\widetilde{\psi}, \theta_0, \phi)\right)\, d\phi d\widetilde{\psi} \\
&= \int_0^{\psi}\int_0^{2\pi} -\partial_\phi v\left(\mathbf{r}(\widetilde{\psi}, \theta_0, \phi)\right)\, d\phi d\widetilde{\psi} \\
&= \int_0^{\psi}\int_0^{2\pi} -h(\widetilde{\psi})\, d\theta d\widetilde{\psi} \text{ as } \lambda \text{ is periodic} \\
&= -2\pi\int_0^{\psi} h(\widetilde{\psi})\, d\widetilde{\psi}
\end{aligned}$$

$$\begin{cases} \iota(\psi) = \frac{d\Psi_P(\psi)/d\psi}{d\Psi_T(\psi)/d\psi} \\ \Psi_T(\psi) = 2\pi\psi \end{cases} \Rightarrow d\Psi_P(\psi)/d\psi = 2\pi\iota(\psi)$$

$$\Rightarrow h(\psi) = -\iota(\psi)$$

Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Poloidal flux $\Rightarrow h(\psi) = -\iota(\psi)$

Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Toroidal current $\Rightarrow j(\psi) = 1$
- Poloidal current $\Rightarrow h(\psi) = -\iota(\psi)$

Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
$$v(\psi, \theta, \phi) = j(\psi)\theta + h(\psi)\phi + \lambda(\psi, \theta, \phi)$$
- Toroidal current $\Rightarrow j(\psi) = 1$
- Poloidal current $\Rightarrow h(\psi) = -\iota(\psi)$
- Magnetic field

$$\begin{aligned}\boldsymbol{B} &= \nabla\psi \times \nabla\Big(\theta - \iota\phi + \lambda\Big) \\ &= \nabla\psi \times \nabla\Big(\theta + \lambda\Big) - \iota\nabla\psi \times \nabla\phi \text{ since } \nabla(\iota(\psi)) = \frac{d\iota(\psi)}{d\psi}\nabla\psi(\psi, \theta, \phi)\end{aligned}$$

Magnetic coordinates : Simplified contravariant form

- $\boldsymbol{B} = \nabla\psi \times \nabla v$ and $\exists\ (j, h, \lambda)$ with λ periodic wrt (θ, ϕ) s.t.
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Definition of Magnetic coordinates

- $(\psi, \vartheta, \varphi)$ with $\vartheta = \theta + \lambda$, $\varphi = \phi$
$$\boldsymbol{B}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))} = \underbrace{[\nabla\psi \times \nabla\vartheta]}_{g^{-1/2}\partial_\varphi \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))} - \iota(\psi)\underbrace{[\nabla\psi \times \nabla\varphi]}_{-g^{-1/2}\partial_\vartheta \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))}$$

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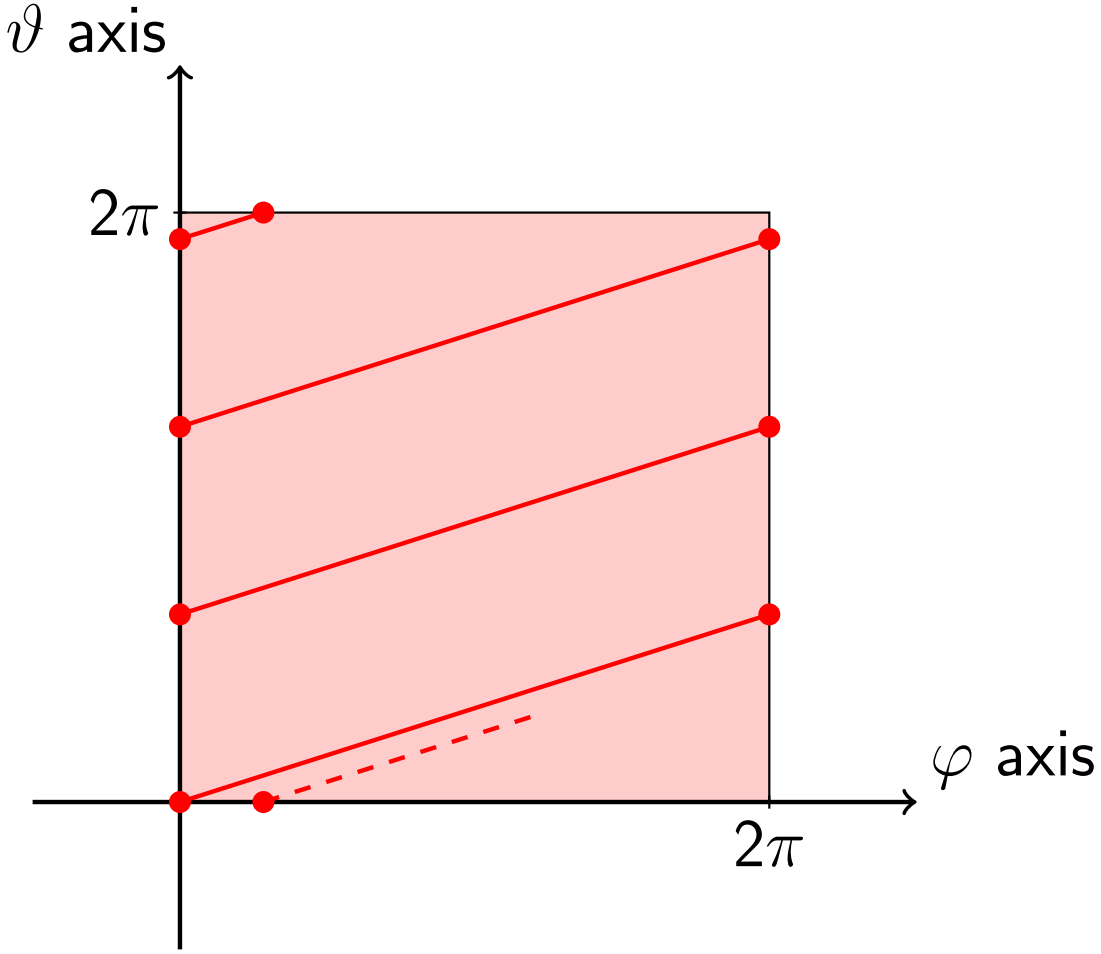
$$\boldsymbol{B}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))} = \underbrace{[\nabla\psi \times \nabla\vartheta]}_{g^{-1/2}\partial_\varphi \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))} - \iota(\psi)\underbrace{[\nabla\psi \times \nabla\varphi]}_{-g^{-1/2}\partial_\vartheta \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))}$$

$$\boxed{\text{Field lines are straight in magnetic coordinates : } \iota = \frac{\boldsymbol{B}\cdot\nabla\vartheta}{\boldsymbol{B}\cdot\nabla\varphi}}$$

Field lines in magnetic coordinates



Field lines are straight in magnetic coordinates : $\iota = \frac{\boldsymbol{B}\cdot\nabla\vartheta}{\boldsymbol{B}\cdot\nabla\varphi}$

Boozer coordinates

Magnetic field strength

Magnetic vs Boozer coordinates

Magnetic	Boozer
$\nabla \cdot \boldsymbol{B} = 0$	$\nabla \cdot \boldsymbol{B} = 0$ $\boldsymbol{J} \cdot \nabla\psi = 0$
$\boldsymbol{B} = \nabla\psi \times \nabla\vartheta$ $-\iota(\psi)\nabla\psi \times \nabla\varphi$	$\boldsymbol{B} = \nabla\psi \times \nabla\vartheta_B$ $-\iota(\psi)\nabla\psi \times \nabla\varphi_B$
$\boldsymbol{B} = B_\vartheta(\psi,\vartheta,\varphi)\nabla\vartheta + B_\varphi(\psi,\vartheta,\varphi)\nabla\varphi$ $+B_\psi(\psi,\vartheta,\varphi)\nabla\psi$	$\boldsymbol{B} = I(\psi)\nabla\vartheta_B + G(\psi)\nabla\varphi_B$ $+\ K(\psi,\vartheta_B,\varphi_B)\nabla\psi$ $G(\psi) = \mu_0 I_P(\psi)/2\pi$ $I(\psi) = \mu_0 I_T(\psi)/2\pi$

Jacobian	
$\dfrac{B_\varphi(\psi,\vartheta,\varphi) + \iota(\psi)B_\vartheta(\psi,\vartheta,\varphi)}{B^2(\psi,\vartheta,\varphi)}$	$\dfrac{G(\psi) + \iota(\psi)I(\psi)}{B^2(\psi,\vartheta_B,\varphi_B)}$