The Magnetic Differential Equation

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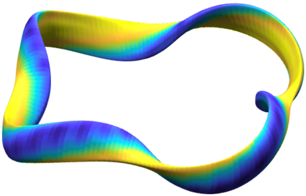
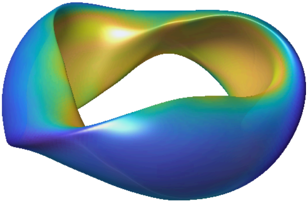
An Introduction to Stellarators (arXiv :1908.05360)
From magnetic fields to symmetries and optimization

- Self-contained and introductory
- Scope : challenges in stellarator design
- Goal : Stimulate cross-disciplinary collaboration





Today

- The magnetic differential equation
- Related singularities & interpretation

Reminders

Fundamental simplifying assumption

- Existence of toroidal nested flux surfaces
- Flux labels and flux coordinates

Magnetic coordinates

- $\nabla \cdot \mathbf{B} = 0$
- $\boldsymbol{B}_{\,(\boldsymbol{r}(\psi,\vartheta,\varphi))} = \underbrace{[\nabla\psi \times \nabla\vartheta]}_{g^{-1/2}\partial_\varphi \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))} - \iota(\psi)\underbrace{[\nabla\psi \times \nabla\varphi]}_{-g^{-1/2}\partial_\vartheta \mathbf{r}}{}_{(\boldsymbol{r}(\psi,\vartheta,\varphi))}$
- Field lines are straight in magnetic coordinates

Boozer coordinates

Magnetic coordinates : simplified covariant form

Equilibrium magnetic field

- Pressure profile $p(\psi) \Rightarrow \nabla p = dp(\psi)/d\psi \nabla\psi$
- Momentum conservation $\boldsymbol{J} \times \boldsymbol{B} = \nabla p \Rightarrow \boldsymbol{J} \cdot \nabla\psi = 0$
- Ampere's law $\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} \Rightarrow \nabla \times \boldsymbol{B} \cdot \nabla\psi = 0$

- $\nabla \times \boldsymbol{B}$ in the contravairant basis, leveraging duality
- $\partial_\vartheta B_\varphi - \partial_\varphi B_\vartheta = 0 \Rightarrow \exists\ H$ s.t. $\begin{cases} \partial_\vartheta H = B_\vartheta + I(\psi,\varphi) \\ \partial_\varphi H = B_\varphi + G(\psi,\vartheta) \end{cases}$

$$\Rightarrow \begin{cases} 0 = \int_0^{2\pi} [\partial_\vartheta B_\varphi - \partial_\varphi B_\vartheta](\psi_0,\vartheta_0,\varphi')d\varphi' \Rightarrow \partial_\vartheta G = 0 \\ 0 = \int_0^{2\pi} [\partial_\vartheta B_\varphi - \partial_\varphi B_\vartheta](\psi_0,\vartheta_0,\varphi')d\vartheta' \Rightarrow \partial_\varphi I = 0 \end{cases}$$

- *Define $K := B_\psi - \partial_\psi H$

$$\boldsymbol{B}\ _{(\boldsymbol{r}(\psi,\vartheta,\varphi))} = \nabla H\ _{(\boldsymbol{r}(\psi,\vartheta,\varphi))}$$
$$+I(\psi)\nabla\vartheta\ _{(\boldsymbol{r}(\psi,\vartheta,\varphi))} + G(\psi)\nabla\varphi\ _{(\boldsymbol{r}(\psi,\vartheta,\varphi))} + K(\psi,\vartheta,\varphi)\nabla\psi\ _{(\boldsymbol{r}(\psi,\vartheta,\varphi))}$$

Boozer coordinates Choice of angles (ϑ_B, φ_B) to eliminate H

- $\boldsymbol{B} = I(\psi)\nabla\vartheta_B + G(\psi)\nabla\varphi_B + K(\psi,\vartheta_B,\varphi_B)\nabla\psi$

Covariant components in Boozer coordinates

- $\boldsymbol{B} = I\nabla\vartheta_B + G\nabla\varphi_B + K\nabla\psi$

Covariant components I and G

- Related to poloidal and toroidal currents

Equation for the radial covariant component K

- $\sqrt{g}\mathbf{B} = \partial_{\varphi_B}\mathbf{r} + \iota\partial_{\vartheta_B}\mathbf{r}$
- $\sqrt{g}\nabla\times\mathbf{B} = (I' - \partial_{\vartheta_B}K)\partial_{\varphi_B}\mathbf{r} + (\partial_{\varphi_B}K - G')\partial_{\vartheta_B}\mathbf{r}$

$\Rightarrow$ $(\nabla\times\mathbf{B})\times\mathbf{B} = \frac{\partial_{\varphi_B}K + \iota\partial_{\vartheta_B}K - G' - \iota I'}{\sqrt{g}}\nabla\psi$

MHD equilibrium + pressure profile $p(\psi)$

- $(\nabla\times\mathbf{B})\times\mathbf{B} = \mu_0 p'\nabla\psi$

$\Rightarrow$ $\partial_{\varphi_B}K + \iota\partial_{\vartheta_B}K = G' + \iota I' + \sqrt{g}\mu_0 p'$

Covariant components in Boozer coordinates

- $\boldsymbol{B} = I\nabla\vartheta_B + G\nabla\varphi_B + K\nabla\psi$

Covariant components I and G

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$\Rightarrow$ $(\nabla\times\mathbf{B})\times\mathbf{B} = \frac{\partial_{\varphi_B}K + \iota\partial_{\vartheta_B}K - G' - \iota I'}{\sqrt{g}}\nabla\psi$

MHD equilibrium + pressure profile $p(\psi)$

- $(\nabla\times\mathbf{B})\times\mathbf{B} = \mu_0 p'\nabla\psi$

$\Rightarrow$ $\partial_{\varphi_B}K(\psi,\vartheta_B,\varphi_B) + \iota(\psi)\partial_{\vartheta_B}K(\psi,\vartheta_B,\varphi_B)$

$$= G'(\psi) + \iota(\psi)I'(\psi) + \mu_0\sqrt{g}(\psi,\vartheta_B,\varphi_B)p'(\psi)$$

Magnetic vs Boozer coordinates

Magnetic	Boozer
$\nabla \cdot \boldsymbol{B} = 0$	$\nabla \cdot \boldsymbol{B} = 0$ $\boldsymbol{J} \cdot \nabla\psi = 0$
$\boldsymbol{B} = \nabla\psi \times \nabla\vartheta$ $-\iota(\psi)\nabla\psi \times \nabla\varphi$	$\boldsymbol{B} = \nabla\psi \times \nabla\vartheta_B$ $-\iota(\psi)\nabla\psi \times \nabla\varphi_B$
$\boldsymbol{B} = B_\vartheta(\psi,\vartheta,\varphi)\nabla\vartheta + B_\varphi(\psi,\vartheta,\varphi)\nabla\varphi$ $+B_\psi(\psi,\vartheta,\varphi)\nabla\psi$	$\boldsymbol{B} = I(\psi)\nabla\vartheta_B + G(\psi)\nabla\varphi_B$ $+ K(\psi,\vartheta_B,\varphi_B)\nabla\psi$ $G(\psi) = \mu_0 I_P(\psi)/2\pi$ $I(\psi) = \mu_0 I_T(\psi)/2\pi$

Jacobian	
$\dfrac{B_\varphi(\psi,\vartheta,\varphi) + \iota(\psi)B_\vartheta(\psi,\vartheta,\varphi)}{B^2(\psi,\vartheta,\varphi)}$	$\dfrac{G(\psi) + \iota(\psi)I(\psi)}{B^2(\psi,\vartheta_B,\varphi_B)}$

The Magnetic Differential Equation

$$\partial_{\varphi_B} K(\psi, \vartheta_B, \varphi_B) + \iota(\psi)\partial_{\vartheta_B} K(\psi, \vartheta_B, \varphi_B)$$
$$= G'(\psi) + \iota(\psi) I'(\psi) + \mu_0 \sqrt{g}(\psi, \vartheta_B, \varphi_B) p'(\psi)$$
$$(\vartheta_B, \varphi_B) \in [0, 2\pi]^2$$

Differential operator
&
existence of solutions

Differential operator in the two-dimensional setting I

- $\mathcal{D} := \partial_{\varphi_B} + \iota \partial_{\vartheta_B}$ with $\psi =$const.

Integral on the doubly periodic domain

- Periodic function u of two variables :
- $$\int_0^{2\pi}\int_0^{2\pi} \partial_{\varphi_B} u(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = \int_0^{2\pi} u(\vartheta_B, 2\pi) - u(\vartheta_B, 0)\, d\vartheta_B = 0 \text{ by periodicity}$$
- $$\int_0^{2\pi}\int_0^{2\pi} \partial_{\vartheta_B} u(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = \int_0^{2\pi} u(2\pi, \varphi_B) - u(0, \varphi_B)\, d\varphi_B = 0 \text{ by periodicity}$$

$$\Rightarrow \int_0^{2\pi}\int_0^{2\pi} \mathcal{D} u(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0$$

About existence of solutions to the MDE I

- $\mathcal{D} := \partial_{\varphi_B} + \iota\partial_{\vartheta_B}$ with ψ =const.

Integral on the doubly periodic domain

- Periodic function u of two variables :
- $\int_0^{2\pi}\int_0^{2\pi} \mathcal{D}u(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0$

Integral of the MDE

- Periodic solution K

$$\iota\frac{\partial K}{\partial\vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial\varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

- Necessary condition for existence of doubly-periodic solutions

$$\boxed{\int_0^{2\pi}\int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0}$$

Differential operator in the two-dimensional setting II

- $\mathcal{D} := \partial_{\varphi_B} + \iota \partial_{\vartheta_B}$ with ψ =const.

Derivative along any curve $s \mapsto (\varphi_P(s), \vartheta_P(s))$

- Function u of two variables : $s \mapsto u(\varphi_P(s), \vartheta_P(s))$

$$\frac{d}{ds}\left[u(\varphi_P(s), \vartheta_P(s))\right]$$
$$= \varphi'_P(s)\frac{\partial u}{\partial \varphi_B}(\varphi_P(s), \vartheta_P(s)) + \vartheta'_P(s)\frac{\partial u}{\partial \vartheta_B}(\varphi_P(s), \vartheta_P(s))$$
$$= \left(\varphi'_P(s)\frac{\partial}{\partial \varphi_B} + \vartheta'_P(s)\frac{\partial}{\partial \vartheta_B}\right) u\big(\varphi_P(s), \vartheta_P(s)\big)$$

Along a particular type of curve

- Assume $\varphi'_P(s) = 1$ and $\vartheta'_P(s) = \iota$

$$\frac{d}{ds}\left[u(\varphi_P(s), \vartheta_P(s))\right] = \mathcal{D}u\big(\varphi_P(s), \vartheta_P(s)\big)$$

- **Characteristic curves**

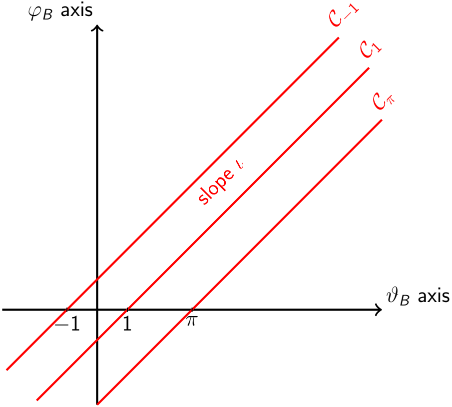
Characteristic curves

$$\begin{cases} \varphi_P'(s) = \iota, \\ \vartheta_P'(s) = 1, \end{cases}$$

Define $\mathcal{C}_\alpha$

$$\begin{cases} \varphi_P(s) = \iota s, \\ \vartheta_P(s) = s + \alpha, \end{cases}$$

- **Straight lines** in the (φ_B, ϑ_B)-plane



PDE reduction to an ODE along each characteristic

$$\left(\partial_{\varphi_B} + \iota \partial_{\vartheta_B}\right) K(\varphi_B, \vartheta_B) = F(\varphi_B, \vartheta_B)$$

$$\Rightarrow \frac{d}{ds} \left[K(\varphi_P(s), \vartheta_P(s))\right] = F(\varphi_P(s), \vartheta_P(s))$$

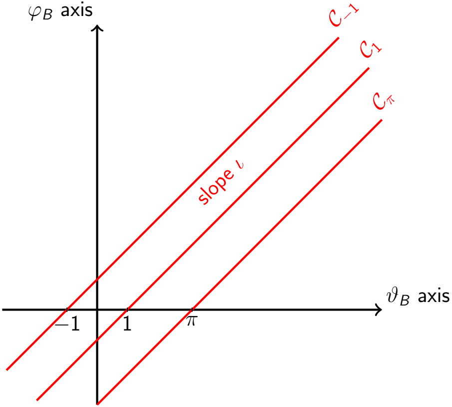
Characteristic curves

$$\begin{cases} \varphi_P'(s) = \iota, \\ \vartheta_P'(s) = 1, \end{cases}$$

Define $\mathcal{C}_\alpha$

$$\begin{cases} \varphi_P(s) = \iota s, \\ \vartheta_P(s) = s + \alpha, \end{cases}$$

- **Straight lines** in the (φ_B, ϑ_B)-plane
- Just like magnetic field lines



PDE reduction to an ODE along each characteristic

$$\left(\partial_{\varphi_B} + \iota \partial_{\vartheta_B}\right) K(\varphi_B, \vartheta_B) = F(\varphi_B, \vartheta_B)$$

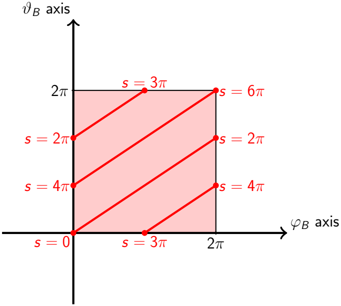
$$\Rightarrow \frac{d}{ds}\left[K(\varphi_P(s), \vartheta_P(s))\right] = F(\varphi_P(s), \vartheta_P(s))$$

Characteristics and doubly periodic setting

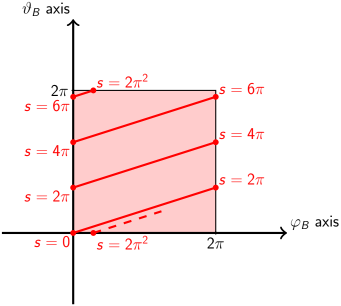
Given F periodic, we search for K periodic such that :

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

Rational and irrational characteristics



$\iota = 2/3$



$\iota = 1/\pi$

Closed field lines $\iota \in \mathbb{Q}$

Infinite field lines $\iota \notin \mathbb{Q}$

About existence of solutions to the MDE II $\iota \in \mathbb{Q}$

- $\mathcal{D} := \partial_{\varphi_B} + \iota \partial_{\vartheta_B}$ with ψ =const.
- Any closed charateristic $\mathcal{C}_\alpha$: $\forall s \in [0, s_D], \begin{cases} \varphi_P(s) = \iota s \\ \vartheta_P(s) = s + \alpha \end{cases}$

Integral along any closed characteristic

- Periodic function u of two variables
- $\displaystyle\int_0^{s_D} \mathcal{D}u(\vartheta_P(s), \varphi_P(s))\, ds = \int_0^{s_D} \frac{d}{ds}\left[u(\varphi_P(s), \vartheta_P(s))\right] ds = 0$

Integral of the MDE along closed characteristics

- Periodic solution K

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

- Necessary condition for existence of doubly-periodic solutions

$$\boxed{\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0 \quad \forall \mathcal{C}_\alpha}$$

A comment on the rational case

Periodic solutions along closed characteristics $\iota =^{*} \frac{N}{D} \in \mathbb{Q}$

- Change of variable $s_D = 2\pi D$

$$\begin{array}{ccc} [0, 2\pi D] \times [0, 2\pi/N) & \rightarrow & [0, 2\pi]^2 \\ (s, \alpha) & \mapsto & (\vartheta_P(s), \varphi_P(s)) = (\iota s, s + \alpha) \end{array}$$

$$\int_0^{2\pi} \int_0^{2\pi} F(\varphi_B, \vartheta_B)\, d\varphi_B \; d\vartheta_B = \iota \int_0^{2\pi/N} \left(\int_0^{2\pi D} F(\iota s, s + \alpha)\, ds \right) d\alpha$$

- $\displaystyle\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0, \quad \forall \alpha$

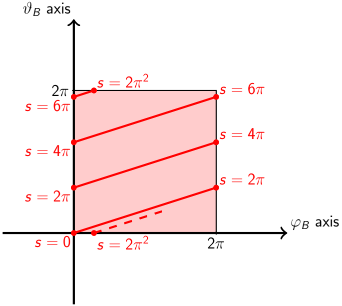
$$\Rightarrow \int_0^{2\pi} \int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0$$

About existence of solutions to the MDE II $\iota \notin \mathbb{Q}$

- $\mathcal{D} := \partial_{\varphi_B} + \iota \partial_{\vartheta_B}$ with ψ =const.
- Any infinite charateristic $\mathcal{C}_\alpha$: $\forall s \in \mathbb{R}, \begin{cases} \varphi_P(s) = \iota s \\ \vartheta_P(s) = s + \alpha \end{cases}$

Integral along any open characteristic

- Periodic function u of two variables
- $\displaystyle\int_{\mathbb{R}} \mathcal{D}u(\vartheta_P(s), \varphi_P(s))\, ds = \int_{\mathbb{R}} \frac{d}{ds}\left[u(\varphi_P(s), \vartheta_P(s))\right] ds$



So what if $\iota \notin \mathbb{Q}$?

Summary of necessary conditions for existence of periodic solutions

The Magnetic Differential Equation

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

Doubly-periodic solutions

- $\iota \notin \mathbb{Q}$
- $\boxed{\int_0^{2\pi} \int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0}$

Periodic solutions along closed characteristics

- $\iota \in \mathbb{Q} \Leftrightarrow$ closed characteristics
- $\boxed{\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0, \quad \forall \mathcal{C}\alpha}$

Construction of solutions

Fourier context

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

- Assuming F is doubly-periodic and square integrable
- Fourier expansion

$$F(\vartheta_B, \varphi_B) = \sum_{(m,n) \in \mathbb{Z}^2} b_{m,n} e^{i(m\vartheta_B - n\varphi_B)},$$

- Seek the solution K as a Fourier series

$$K(\vartheta_B, \varphi_B) = \mathcal{K} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} a_{m,n} e^{i(m\vartheta_B - n\varphi_B)}.$$

Interpretation of the necessary condition ?

Necessary condition in the Fourier context $\iota \notin \mathbb{Q}$

- Fourier expansion of the doubly-periodic right hand side F

$$F(\vartheta_B, \varphi_B) = \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} e^{i(m\vartheta_B - n\varphi_B)},$$

Doubly-periodic solutions

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B \\
&= \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} \left(\underbrace{\int_0^{2\pi} e^{im\vartheta_B}\, d\vartheta_B}_{=2\pi\delta(m)} \right) \left(\underbrace{\int_0^{2\pi} e^{-in\varphi_B}\, d\varphi_B}_{=2\pi\delta(n)} \right) \\
&= b_{0,0}
\end{aligned}$$

- $\int_0^{2\pi}\int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0 \Leftrightarrow b_{0,0} = 0$

Necessary condition in the Fourier context $\iota \in \mathbb{Q}$

- Fourier expansion of the doubly-periodic right hand side F

$$F(\vartheta_B, \varphi_B) = \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} e^{i(m\vartheta_B - n\varphi_B)},$$

Periodic solutions along closed characteristics

$$\int_0^{s_D} F(\iota s, s+\alpha)\, ds \;=\; \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} \left(\underbrace{\int_0^{2\pi D} e^{i(m\iota - n)s}\, ds}_{2\pi D \delta(n-\iota m)} \right) e^{-in\alpha}$$

- $\displaystyle\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0, \quad \forall \mathcal{C}_\alpha$
  $\Leftrightarrow b_{m,n} = 0 \ \forall (m,n) \in \mathbb{Z}^2$ such that $m\iota = n$

Summary of necessary conditions - Fourier context

The Magnetic Differential Equation

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

Doubly-periodic solutions $\iota \notin \mathbb{Q}$

$$\int_0^{2\pi} \int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0 \Leftrightarrow b_{0,0} = 0$$

Periodic solutions along closed characteristics $\iota \in \mathbb{Q}$

$$\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0, \forall \mathcal{C}_\alpha \Leftrightarrow b_{m,n} = 0\, \forall (m,n) \in \mathbb{Z}^2 \text{ s.t. } m\iota = n$$

Single statement ?

Summary of necessary conditions - Fourier context

The Magnetic Differential Equation

$$\iota \frac{\partial K}{\partial \vartheta_B}(\vartheta_B, \varphi_B) + \frac{\partial K}{\partial \varphi_B}(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

Doubly-periodic solutions $\iota \notin \mathbb{Q}$

$$\int_0^{2\pi} \int_0^{2\pi} F(\vartheta_B, \varphi_B)\, d\vartheta_B d\varphi_B = 0 \Leftrightarrow b_{0,0} = 0$$

Periodic solutions along closed characteristics $\iota \in \mathbb{Q}$

$$\int_0^{s_D} F(\vartheta_P(s), \varphi_P(s))\, ds = 0, \forall \mathcal{C}_\alpha \Leftrightarrow b_{m,n} = 0\, \forall (m,n) \in \mathbb{Z}^2 \text{ s.t. } m\iota = n$$

Single statement ? $\boxed{b_{m,n} = 0\ \forall (m,n) \in \mathbb{Z}^2 \text{ s.t. } m\iota = n}$

Towards a Fourier solution - Assuming convergence

$$K(\vartheta_B, \varphi_B) = \mathcal{K} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} a_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\partial_{\varphi_B} K(\vartheta_B, \varphi_B) + \iota \partial_{\vartheta_B} K(\vartheta_B, \varphi_B) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} i(-n + \iota m) a_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\partial_{\varphi_B} K(\vartheta_B, \varphi_B) + \iota \partial_{\vartheta_B} K(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

$$\Leftrightarrow \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} i(-n+\iota m) a_{m,n} e^{i(m\vartheta_B - n\varphi_B)} = \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\Leftrightarrow a_{m,n} i(\iota m - n) = b_{m,n} \quad \forall (m,n) \in \mathbb{Z}^2$$

Comments ?

Towards a Fourier solution - Assuming convergence

$$K(\vartheta_B, \varphi_B) = \mathcal{K} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} a_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\partial_{\varphi_B} K(\vartheta_B, \varphi_B) + \iota \partial_{\vartheta_B} K(\vartheta_B, \varphi_B) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} i(-n + \iota m) a_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\partial_{\varphi_B} K(\vartheta_B, \varphi_B) + \iota \partial_{\vartheta_B} K(\vartheta_B, \varphi_B) = F(\vartheta_B, \varphi_B)$$

$$\Leftrightarrow \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} i(-n+\iota m) a_{m,n} e^{i(m\vartheta_B - n\varphi_B)} = \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} e^{i(m\vartheta_B - n\varphi_B)}$$

$$\Leftrightarrow a_{m,n} i(\iota m - n) = b_{m,n} \quad \forall (m,n) \in \mathbb{Z}^2$$

Comments ?

- $\forall (m,n) \in \mathbb{Z}^2$ s.t. $m\iota = n$, $a_{m,n} i(\iota m - n) = 0$
- $\forall (m,n) \in \mathbb{Z}^2$ s.t. $m\iota = n$, $b_{m,n}$? $a_{m,n}$?

A general Fourier solution

$$a_{m,n} i(\iota m - n) = b_{m,n} \quad \forall (m,n) \in \mathbb{Z}^2$$

Under the condition $\forall (m,n) \in \mathbb{Z}^2$ s.t. $m\iota = n$, $b_{m,n} = 0$

- $\forall (m,n) \in \mathbb{Z}^2$ s.t. $m\iota = n$, $a_{m,n}$ is free

A general solution

$$K(\vartheta_B, \varphi_B) = \mathcal{K} + \sum_{(m,n) \neq (0,0)} \left(\frac{i b_{m,n}}{n - \iota m} + \Delta_{mn} \delta(n - \iota m) \right) e^{i(m\vartheta_B - n\varphi_B)}$$

- Convergence
- Numerical approximation

Interpretation of $\Delta_{m,n}s$ in terms of characteristics

$\mathcal{S} := \{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)\}$

$\iota = \dfrac{N}{D}$ rational	ι irrational
$F(x_1, x_2) = \sum_{(m,n)\in\mathcal{S}} b_{m,n} e^{i(mx_1 - nx_2)}$	
$b_{m,n} = 0 \ \forall (m,n) \in \mathbb{Z}^2$ s.t. $n = \iota m$	$b_{0,0} = 0$
$\mathcal{K}$ $\left\{\Delta_{m,n}, (m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}, \iota = \dfrac{n}{m}\right\}$	$\mathcal{K}$
$u(x_1, x_2) = \mathcal{K}$ $+ \sum_{(m,n)\in\mathcal{S}} \Delta_{mn} \delta(n - \iota m) e^{i(mx_1 - nx_2)}$ $+ \sum_{(m,n)\in\mathcal{S}} \dfrac{i b_{m,n}}{n - \iota m} e^{i(mx_1 - nx_2)}$	$u(x_1, x_2)$ $= \mathcal{K}$ $+ \sum_{(m,n)\in\mathcal{S}} \dfrac{i b_{m,n}}{n - \iota m} e^{i(mx_1 - nx_2)}$

A general Fourier solution and *singularities*

$$K(\vartheta_B, \varphi_B) = \mathcal{K} + \sum_{(m,n)\neq(0,0)} \left(\frac{i b_{m,n}}{n - \iota m} + \Delta_{mn} \delta(n - \iota m) \right) e^{i(m\vartheta_B - n\varphi_B)}$$

So-called singularities

- "$1/x$" singularity
  $\forall (m,n) \in \mathbb{Z}^2$ such that $\iota = n/m$, $b_{m,n} = 0$
  $\leftrightarrow$ (non) existence
- "δ" singularity
  free parameters $\{\Delta_{m,n}, (m,n) \in \mathbb{Z}^2, \iota = n/m\}$
  $\leftrightarrow$ (non) uniqueness

Comments ?

- ι rational
- ι irrational

The "$1/x$" *singularity*

$$K(\vartheta_B,\varphi_B) = \mathcal{K} + \sum_{(m,n)\neq(0,0)} \left(\frac{ib_{m,n}}{n-\iota m} + \Delta_{mn}\delta(n-\iota m)\right) e^{i(m\vartheta_B - n\varphi_B)}$$

$F(\vartheta_B,\varphi_B) = G' + \iota I' + \mu_0\sqrt{g}(\vartheta_B,\varphi_B)p'$

- $\sqrt{g}(\vartheta_B,\varphi_B) = c + \sum_{(m,n)\neq(0,0)} c_{m,n}e^{i(m\vartheta_B - n\varphi_B)}$
- $b_{0,0} = G' + \iota I' + \mu_0 cp'$ & $b_{m,n} = \mu_0 p' c_{m,n}$

"$1/x$" singularity

- $\forall(m,n)\in\mathbb{Z}^2$ such that $\iota = n/m$, $b_{m,n} = 0$
  $\Leftrightarrow \forall(m,n)\in\mathbb{Z}^2$ such that $\iota = n/m$, $p'c_{m,n} = 0$
- p plasma pressure

Back to the 3D setting

$$F(\psi,\vartheta_B,\varphi_B) = G'(\psi) + \iota(\psi)I'(\psi) + \mu_0\sqrt{g}(\psi,\vartheta_B,\varphi_B)p'(\psi)$$

The 3D perspective* & singularities

$$\partial_{\varphi_B} K(\psi, \vartheta_B, \varphi_B) + \iota(\psi)\partial_{\vartheta_B} K(\psi, \vartheta_B, \varphi_B)$$

$$= \mu_0(\sqrt{g}(\psi, \vartheta_B, \varphi_B) - c(\psi))p'(\psi)$$

MDE for the radial covariant comp. of the magnetic field

The rotational transform ι is a **flux function**

$$K(\psi, \vartheta_B, \varphi_B) = \underbrace{\mathcal{K}(\psi)}$$

$$+ \sum_{(m,n)\neq(0,0)} \left(\frac{i\mu_0 p'(\psi)}{n - \iota(\psi) m} c_{m,n}(\psi) + \underbrace{\Delta_{mn}(\psi)} \delta(n - \iota(\psi) m) \right) e^{i(m\vartheta_B - n\varphi_B)}$$

Singularities at rational flux surfaces $\iota(\psi) \in \mathbb{Q}$

"$1/x$" singularity - p plasma pressure

- $\forall (m, n) \in \mathbb{Z}^2$ such that $\iota(\psi) = n/m$, $p'(\psi) c_{m,n}(\psi) = 0$
  - If $p'(\psi) = 0$ for all ψ such that $\iota(\psi)$ is rational then the condition is satisfied

"δ" singularity

- free parameters $\{\Delta_{m,n}(\psi), (m, n) \in \mathbb{Z}^2, \iota(\psi) = n/m\}$

MDE for the parallel current density

For comparison

$$\partial_{\varphi_B} K(\psi, \vartheta_B, \varphi_B) + \iota(\psi)\partial_{\vartheta_B} K(\psi, \vartheta_B, \varphi_B)$$
$$= \mu_0(\sqrt{g}(\psi, \vartheta_B, \varphi_B) - c(\psi))p'(\psi)$$

MDE for the parallel current density

$$\partial_\varphi \left(\frac{J_{||}}{B}\right) + \iota \partial_\vartheta \left(\frac{J_{||}}{B}\right) = -\sqrt{g}\nabla \cdot \frac{\boldsymbol{B} \times \nabla p}{B^2}$$

Physical current $I_P = \int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}}\, d^2x$

- "δ" singularity
  Current sheet on rational surfaces ⇒ physical ✔
- "$1/x$" singularity
  Pfirsch-Schluter current ⇒ non-physical (infinite I_P)✘

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For comparison

$$\partial_{\varphi_B} K(\psi, \vartheta_B, \varphi_B) + \iota(\psi)\partial_{\vartheta_B} K(\psi, \vartheta_B, \varphi_B)$$

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Physical current $I_P = \int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}}\, d^2x$

- "δ" singularity related to non-uniqueness
  Current sheet on rational surfaces ⇒ physical ✔
- "$1/x$" singularity related to non-existence
  Pfirsch-Schluter current ⇒ non-physical (infinite I_P)✘

To satisfy the existence condition

- ☐ Assumption on the geometry of the field (g)
- ☒ Assumption on the pressure profile

Constant pressure

- $p'(\psi) = 0$, $\nabla p = \mathbf{0}$
- Limited physical interest

Piece-wise constant profile

- Finite number of interface supporting pressure jumps
- Separating regions of constant pressure

Fractal profile

- (Hudson '17) Theory and discretization

More or less assumptions

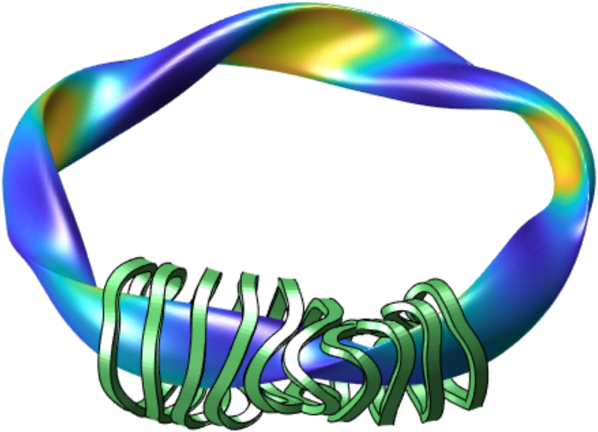
Existence condition

- Necessary condition for the existence of solutions
- Condition on the right hand side of the PDE
  - Assumption on the geometry of the field (g)
  - Assumption on the pressure profile

Non-existence of solutions

- Back to the derivation of the PDE
  - Assumption of existence of toroidal nested flux surfaces
- Interpretation of current singularities at rational surfaces
- Allowing magnetic islands
  - Finite number of nested flux surfaces
    MRxMHD model, SPEC code
  - No assumption on flux surfaces

Thank you.



Last closed magnetic surface of the W7-X configuration

Electromagnetic coil shapes for one field period of the device