An Introduction to Plasma Turbulence

Now that we have visited some of the important neutral fluid turbulence topics, we can now re-examine some of the same concepts for plasmas. Since we are now going to couple the velocity field to $E$ and $B$, things are obviously going to get messier in some ways, but it turns out that it also becomes simpler in others.

Recall that in hydro turbulence,

- Scale invariance and
- Local, i.e., energy transfer

\[ \Rightarrow E \sim \frac{\delta u^2}{\tau_e} \sim \text{const.} \]

Further, there was only one time scale in the system, the eddy turnover time

\[ \tau_e = \frac{e}{\delta u_e} \Rightarrow E \sim \frac{\delta u^2}{\tau_e} \sim \text{const.} \quad \therefore \text{Kolmogorov 4/3 energy spectrum from simple dimensional analysis.} \]

Now, let's look at the incompressible MHD case. We'll stick with the incompressible assumption because it makes things simpler, but it also turns out that Alfvén waves (which are incompressible) are the primary component of turbulence.

\[ \rho \left( \frac{\partial \vec{u} + \vec{u} \cdot \nabla \vec{u}}{\rho} \right) = -\nabla \rho + \frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{u} \times \vec{B}) \]  

\[ \frac{\partial \vec{B}}{\partial t} + \vec{u} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{u} + \gamma \frac{\rho}{\rho_{\text{resistivity}}} \vec{B} \]  

where \( \rho = \rho + \frac{\beta_0}{\gamma_{\text{resistivity}}} \) is the total pressure.

If we assume we have a mean magnetic field $\bar{B}_0 = \vec{B}_0 \hat{z}$, we immediately break the isotropy assumption of hydro...
because, unlike a mean flow, a mean \( B_0 \) cannot be transformed away and introduces linear Alfvén waves. So, we now have an inherently anisotropic system w/Alfvén waves.

Let’s re-write eqns (1) and (2) in a more compact form by adding and subtracting them to obtain the Elsasser (1950) eqns:

\[
\partial_t \tilde{\mathbf{E}} = \sqrt{\overline{\mathbf{A}} \cdot \nabla} \tilde{\mathbf{E}} + (\tilde{\mathbf{E}} \cdot \nabla) \tilde{\mathbf{E}} = -\frac{\partial P}{\partial \tilde{\mathbf{E}}} + \frac{\dot{\mathbf{E}}}{2} \nabla \cdot \dot{\mathbf{E}} + \frac{1}{2} \nabla \cdot \dot{\mathbf{E}} + \frac{\dot{\mathbf{E}}}{2} \nabla \cdot \dot{\mathbf{E}}
\]

where \( \tilde{\mathbf{E}} = \mathbf{E} + \frac{\dot{\mathbf{E}}}{\rho_0} \mathbf{A} \quad \mathbf{A} = \mathbf{A} \tilde{\mathbf{E}} \quad \mathbf{B} = \mathbf{B} - \mathbf{B}_0 \)

A few important notes about these eqns:

1) \( \tilde{\mathbf{E}} = 0 \) and \( \tilde{\mathbf{E}}^+ = f(x, y, z + \mathbf{v}_A t) \) or \( \tilde{\mathbf{E}}^- = f(x, y, z - \mathbf{v}_A t) \)

are exact solutions. They represent waves travelling down \( \mathbf{B}_0 \) (\( \tilde{\mathbf{E}}^+ \)) or up \( \mathbf{B}_0 \) (\( \tilde{\mathbf{E}}^- \))

2) The system supports two linear wave modes, both satisfying \( w = k_z \mathbf{v}_A \). These are the Alfvén waves with polarization in the \( \mathbf{E} \times \mathbf{B} \) direction

and \( \mathbf{E} \times (\mathbf{E} \times \mathbf{B}) \) for the pseudo-Alfvén waves, which are the incompressible limit of magnetosonic slow modes. Fast waves are ordered out of the system due to the incompressibility assumption, i.e., \( c_s \to \infty \). It turns out that the slow modes are passive (will show this in a later lecture), so we will focus only on the Alfvén modes.

3) The system is closed. Taking the divergence of \( \tilde{\mathbf{E}} \)

\[
\frac{1}{\rho_0} \nabla \cdot \tilde{\mathbf{E}} = -\nabla \cdot (\tilde{\mathbf{E}}^+ \cdot \nabla \tilde{\mathbf{E}}^+)
\]

4) The non-linear term, \( \tilde{\mathbf{E}}^- \cdot \nabla \tilde{\mathbf{E}}^+ \) requires oppositely propagating Alfvén waves. Further, if we just look at the
behavior of the dot product subject to the constraints above and Fourier transformation

\[ \mathbf{\hat{e}}^+ \times \mathbf{\hat{e}}^- = \mathbf{\hat{k}} \times (\mathbf{\hat{e}}^+ \times \mathbf{\hat{e}}^-) = i \mathbf{\hat{k}} \cdot \mathbf{\hat{e}}^- \mathbf{\hat{k}} \times (\mathbf{\hat{e}}^+ \times \mathbf{\hat{e}}^-) \]

So, the nonlinear term also requires that \( \mathbf{\hat{e}}^+ \) and \( \mathbf{\hat{e}}^- \) have non-zero relative polarization.

Since we now know we are dealing with Alfvénic fluctuations, we know everything is in an Alfvénic state, \( T_A \sim \mathcal{R} \) be scale by scale \( a \) some spectra for \( u \) and \( B \), given \( \mathcal{R} \rightarrow a \), can we now construct the energy spectra for Alfvénic turbulence using these some dimensional arguments as \( K \neq 1 \)?

\[ C_e \sim \frac{\mathcal{R} u_A^2}{T_e} \sim \text{const} \quad \text{is still ok} \]

but now we have 2 choices for \( T_e \)

- eddy turn over time \( T_{eddy} \sim \frac{u_A}{\delta v e} \) (Nonlinear time)
- Alfvén time \( T_A \sim \frac{\mathcal{R} u_A}{v_A} \)

So, which one do we choose and why?

Iroshnikov (1964) - Kraichnan (1965) Theory

To derive an energy spectrum, I'll further assume that the turbulence is weak:

The nonlinear term \( \mathcal{R}^2 \mathcal{R}^2 \mathcal{R}^2 \) \( \ll \) \( \mathcal{R}^2 \mathcal{R}^2 \mathcal{R}^2 \) linear term \( \mathcal{R} \)

This ratio

\[ \frac{\mathcal{R}^2 \mathcal{R}^2 \mathcal{R}^2}{\mathcal{R}^2 \mathcal{R}^2 \mathcal{R}^2} \sim \frac{2}{V_A K_{z1}} \sim \frac{\delta v_A k_{z1}}{V_A k_{z1}} = \frac{\mathcal{R}^2}{V_A^2 k_{z1}} \]

nonlinearity parameter

\( \mathcal{R} \ll 1 \rightarrow \) weak turbulence, linear term dominates

\( \mathcal{R} \gg 1 \rightarrow \) strong turbulence

This feature of turbulence was absent the hydro systems we examined, but it exists when linear
When $\chi \ll 1$, each wave-wave interaction only decorrelates the wave a little. So,

Crossing/interaction time: $\delta t \sim \frac{2\pi}{\nu_A} \sim \chi A$

Change in amplitude: $\delta u \sim \frac{\delta u^2}{\nu_A}$

$\Rightarrow \delta u \sim \frac{\delta u^2}{\nu_A} \delta t \sim \delta u \frac{\delta u l_1}{\nu_A} \sim \delta u \chi$

The “kicks” to the amplitude are random, so they add as

$\sum_{i} \delta u \sim \delta u \chi \sqrt{N}$, where $N = \frac{\delta t}{\chi A}$ is the number of kicks.

The cascade time $\tau_c$ is defined as the time to achieve an order unity change in the amplitude.

So $\frac{\tau_c}{\chi} \sim \delta u \Rightarrow \tau_c \sim \frac{\chi^2}{\epsilon \nu_A}$

$\Rightarrow \delta u \sim (\epsilon \nu_A)^{1/4} \frac{\delta u l_1}{\nu_A}$

We also assumed that $l_1 \sim l_1$ (isotropy).

So, $\mathbb{E} = \mathbb{U} \Rightarrow \delta u \sim (\epsilon \nu_A)^{1/4} \frac{\delta u l_1}{\nu_A}$ or $\mathbb{E} \sim (\epsilon \nu_A)^{1/4} c^{3/2}$

This is the weak interaction self-consistent, i.e., does it hold at all scales in the inertial range?

$\chi \sim \frac{\nu_A}{\nu_A} \frac{\delta u}{\nu_A} \sim \frac{\delta u^2}{\nu_A} \sim (\epsilon \nu_A)^{1/4} \frac{\delta u l_1}{\nu_A}$

$\mathbb{E} \sim \frac{\delta u^2}{\nu_A} \sim \frac{\delta u^4}{\nu_A} \frac{l_1}{l_1} \Rightarrow \chi \sim \frac{\delta u}{\nu_A} \left(\frac{l_1}{l_1}\right)^{1/4}$, which is small for $l \ll l_1$ provided $\delta u < \nu_A$, and it gets smaller with $l$. 

Also, recall that we are dealing with Alfvénic fluctuations, so $\mathbb{E}_B \sim \mathbb{E}_v$. 

Is the weak interaction self-consistent, i.e., does it hold at all scales in the inertial range?
So, IK is self-consistent, and the community rejoiced... until high quality measurements became available. The observed spectra for the magnetic field are closer to $k^{-5/3}$ than $k^{-3/2}$ ($1.66$ and $1.5$ are hard to differentiate). Also, DNSs showed that $\langle l || l \rangle$ is not satisfied.

From Boldyrev et al (2011)
ACE = 10 yrs solar wind data
WIND = 11 yrs

Surfaces of constant $|B|$ measured using UL2300
From Shue et al (2011)
Let's see if we can fix $k$ based on data.

Consider the classic three wave interaction, where waves interact to produce a third.

We have two oppositely propagating Alfvén waves with $w = |k_n V_A$; they must satisfy:

**Energy**: $w(k_1) + w(k_2) = w(k_3) \Rightarrow k_n + k_n 2 = k_n 2$

**Momentum**: $k_1 + k_2 = k_3 \Rightarrow k_n 1 - k_n 2 = k_n 3$

These can only be satisfied if $k_n = 0$, and $k_n 3 = 0$, there is no parallel cascade! And, weak turbulence is mediated by $k_n = 0$ modes.

So, instead of $\propto l_1 \sim l_n$, it should be $\propto l_1 \sim l_n$

and $\delta u_e \sim (E V_A)^{1/4} l_1^{1/2} l_n^{1/2} \sim (E V_A)^{1/4} l_1^{1/2} \Rightarrow E \sim \frac{E V_A}{L} l_1^{-1/2} l_n^{1/2}$

Now, under this new assumption, weak turbulence becomes strong:

$E \sim \frac{\delta u_e}{V_A} \left( \frac{L}{l_1} \right)^{1/2}$, which grows with decreasing $l_1$.

$E \propto 1$ when $l_1 \sim l_{trans} \sim L \left( \frac{\delta u_e}{V_A} \right)^2$

So, at the transition scale, $l_{trans}$, the weak turbulence assumption breaks down and it becomes strong, weakly.

So, let's explore strong turbulence
Let's replace our anisotropy assumption by something less restrictive.

Critical balance: \( \chi \sim 1 \), \( \frac{\partial^2 \rho}{\partial t^2} \times A \sim \rho \frac{\partial^2 \rho}{\partial t^2} \). So, \( L \rho \sim k_n V_A \)

Now, there is just one time scale \( t_e \sim \frac{L_e}{\nu} \)

\[
\xi \sim \frac{\delta u_e^2}{t_e} \sim \frac{\delta u_e^2}{L_e} \Rightarrow \delta u_e \sim (E L_e)^{1/3} \Rightarrow E \sim E^{2/3} k_{\parallel}^{-5/3}
\]

GS Spectrum

Together with \( \chi \sim 1 \)

\[
k_n V_A \sim k_\parallel \delta u_e \sim E^{1/3} k_{\parallel}^{2/3}
\]

\[
\Rightarrow k_{\parallel} \sim \frac{E^{1/3}}{V_A} k_{\parallel}^{2/3}
\]

Reduced parallel cascade

So, GS predict that the wave vector anisotropy grows

Since there is now a parallel cascade, we can also derive a parallel wavenumber spectrum

\[
E \sim \frac{\delta u_e}{t_e} \sim \frac{\delta u_e}{L_e} V_A k_n \Rightarrow \delta u_e \sim (E / V_A) k_{\parallel}^{-1/2}
\]

\[
\Rightarrow E(k_{\parallel}) \sim \frac{E}{V_A} k_{\parallel}^{-2}
\]

End of the inertial range (assumption \( k_0 \) or \( L_n \approx \) kinetic scales)

We now have both viscosity and resistivity that could terminate our inertial range. So, we can construct two Reynolds type numbers
As in hydro
\[ \text{Re} = \frac{\text{convection}}{\text{viscous}} = \frac{\nu}{\nu} \]

Similarly, the magnetic Reynolds number is
\[ \text{Rem} = \frac{\text{convection}}{\text{resistive}} = \frac{\nu}{\nu}, \]

this should not be confused with the Lundquist number \( S := \frac{L \nu}{\nu} \), which
relates the Alfvén crossing time and resistive diffusion.

the magnetic Prandtl number is the ratio of
\[ \text{Prm} = \frac{\text{Rem}}{\text{Re}} = \frac{S}{\nu} \]
and characterizes the relative strength of viscous to magnetic diffusivity.

For simplicity, we'll assume \( \text{Prm} \gg 1 \),
so, at the viscous scale
\[ \nu \sim \frac{\sqrt{\nu}}{l}, \quad \nu \sim \frac{\nu}{l^2/\nu}, \quad \text{and} \quad \nu \sim (\nu l)^{3/4} \]

\[ \Rightarrow l_\nu \sim \nu^{3/4} \] as before weak turbulence result

but \[ \nu^{3/4} = \text{Re} \left( \frac{\nu}{l} \right)^{3/4} \]

\[ \Rightarrow \nu \sim \text{Re}^{-3/4} \left( \frac{\nu}{l} \right)^{3/4} \]

\[ \Rightarrow l_\nu \sim \text{Re}^{-3/4} \left( \frac{\nu}{l} \right)^{3/4} \]

When \( l_\text{trans} \gg l \gg l_\nu \), we have
\[ l_\text{trans} \sim \left( \frac{\nu}{l} \right)^2 \Rightarrow l_\nu \sim \text{Re}^{-3/4} \left( \frac{\nu}{l} \right)^{3/4} \]

So, \( \text{Re} \gg (\nu/l)^{3/4} \) to have strong turbulence and
\[ \Rightarrow \frac{\nu}{\nu} \text{ must be true for weak turbulence} \]

References
2. R. S. Iroshnikov, Sov. Astron. 7, 586 (1964) \( \text{(English translation)} \)