Introduction to Turbulence and \( \mathbb{K4} \)

Turbulence is a chaotic flow regime characterized by diffusivity, rotationality, and dissipation.

Cartoon picture of Turbulence

Mixing of cream (tracer) into coffee

Instead, we stir. Thereby injecting energy at the scale of the cup.

Turbulent diffusion mixes on a shorter timescale.

Equations of Hydrodynamic Turbulence (Order Equs)

1) Incompressibility, \( \rho = \text{const} \implies \nabla \cdot \bar{u} = 0 \)

2) Navier-Stokes (NS)

\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\nabla \left( \frac{\rho}{\rho} \right) + \nu \nabla^2 \bar{u} + \bar{f}
\]

\( \nu \) - Viscosity

\( \bar{f} \) - External forcing

3) Energy: \( \frac{\partial E}{\partial t} + \bar{u} \cdot \nabla E = 0 \)

\( \rho \) = Density

\( \nu \) = Kinematic viscosity

\( E \) = Total internal energy

Aside: When can we assume incompressibility?
1. \( dp/p \ll 1 \): Adiabatic change in \( dp = \left( \frac{\partial p}{\partial p} \right)_s dp \)

Bernoulli's eqn: \( dp + \rho u^2 \) and \( \left( \frac{dp}{dp} \right)_s = \frac{c_s^2}{\gamma} \) \( \Rightarrow \rho u \equiv \gamma c_s^2 \)

\( dp/p \ll 1 \Rightarrow u/c_s = \frac{Ma}{M} \ll 1 \)

\( a) \frac{dp}{dt} \ll c_s \rho v \cdot \nabla : \frac{dp}{dt} \ll \frac{\rho v}{\gamma c_s^2} \) \( \Rightarrow \rho v \cdot \nabla \sim pu/l \)

\( \frac{pu^2}{\gamma c_s^2} \ll \frac{pu}{c_s^2} \) Assumes \( u \ll c \) \( \Rightarrow \frac{pu^2}{\gamma c_s^2} \ll \frac{pu}{c_s^2} \) Information propagates instantaneously

Parameters of the system:

- characteristic (outer-scale) velocity, \( U_0 \) 
- characteristic (outer-scale) length, \( L \)
- viscosity, \( \nu \) set by molecular properties
- Outer-scale is also called the integral or auto-correlation scale

\[ L_{int} \approx \sqrt{\frac{\langle \nu(x,y) \rangle}{\langle \nu(x,y) \rangle}} \]

\[ \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = -\nabla \psi + \nu \nabla^2 u \]

\[ \text{Re} = \frac{U_0 L}{\nu} \]

- When \( Re \) is "small", viscous effects dominate and the flow is linear (laminar)
- When \( Re \) is sufficiently large, the flow becomes chaotic \( \Rightarrow \) turbulent. How large is large enough depends on the system, but values of \( 10^2 \text{--} 10^4 \) are typical.
- The transition between laminar and fully developed turbulence is very messy, so we will focus on fully developed turbulence only.
Phenomenological picture of turbulence

- At each point in the fluid, the velocity is fluctuating around its mean value \( \bar{u}_0 \)
  \[ \bar{u} = \bar{u}_0 + \delta \bar{u} \]

At the outer-scale, \( \delta \bar{u}_0 \sim \delta \bar{u}_L \). Also, we can transform away the mean flow. So, we can redefine \( Re \) in terms of fluctuating quantities
  \[ Re = \frac{\delta u_L L}{\nu} \]

- Let us now consider what happens to the energy in this system
  \[ E = \frac{1}{2} \int \text{d}x \bar{u}^2 \]

Dotting the NS eqn with \( \bar{u} = v \)
  \[ \frac{dE}{dt} = v \int \text{d}x \bar{u} \cdot \delta \bar{u} + \int \text{d}x \bar{u} \cdot \delta \bar{f} \]

If our system is in a stationary state (formally, we are considering the ensemble average \( \langle \cdot \rangle \)) then
  \[ \frac{d\langle E \rangle}{dt} = 0 = v \int \text{d}x \langle \bar{u} \cdot \delta \bar{u} \rangle + \int \text{d}x \langle \bar{u} \cdot \delta \bar{f} \rangle \]

\[ \therefore -v \int \text{d}x \langle \bar{u} \cdot \delta \bar{u} \rangle = \nu \bar{E} \]

In a steady state, the energy input rate, \( \bar{E} \), must match the dissipation rate.

- Let's now construct estimates for various quantities based on dimensional analysis
At the outer-scale, the turbulence is characterized by \( \nu_0, L \). At smaller scales, we can consider the RMS velocity \( u_e \) at scale \( L \), using just the velocity, length scale, and viscosity we now construct other important quantities.

- **Eddy turnover time**: \( t_e \sim \frac{L}{u_e} \)

- At the outer-scale \( t_e \sim \frac{L}{\nu_0} \)

- **Energy injection rate**: \( E \sim \frac{\nu_0^5}{L} \sim \frac{\nu_0^2 L^2}{t_e} \)

Using the energy injection rate, we can re-write \( t_e \sim E^{-\frac{1}{5}} L^{\frac{2}{5}} \) (1)

- **Dissipation time scale**: \( t_{e^*} \sim \frac{\nu}{u_e^2} \) (2)

- **Viscous scale**: Equation (1) and (2)

\[ \Rightarrow \quad \ell_v \sim \left( \frac{\nu^3}{\nu_e^2} \right)^{\frac{1}{4}} \sim L \text{Re}^{-\frac{3}{4}} \ll L \]

Note that this scale has multiple names:

- Viscous scale, inner scale, dissipation scale, Kolmogorov scale

These are all common.

Similarly, the outer-scale is often called the energy containing scale or integral scale.

- Basic picture at this point

\[
\text{Energy injection} \rightarrow \text{energy transport} \rightarrow \text{energy dissipation}
\]

\[ L \gg \ell \gg \ell_v \sim L \text{Re}^{-\frac{3}{4}} \text{ Inertial Range} \]
- Let’s finish the cartoon picture of turbulence before we move on to K41.

- I defined the eddy turnover time above as $t_e \sim \frac{\ell}{u_2}$, but what does this mean? $t_e$ is the characteristic time for a structure of size $\ell$ to undergo a significant distortion due to the relative motion of its components.

$$t_e \sim \frac{\ell}{|v_1 - v_2|}$$

\[ V \cdot u = 0 \implies \text{the area is conserved} \implies \text{if } V \text{ and } u \text{ separate, } 3 \text{ and } 4 \text{ become closer}, \text{i.e. } t_e \text{ is also the typical time for the transfer of energy from scale } \ell \text{ to a smaller scale. Also call this the cascade time.}

\text{Kolmogorov 1941: Turbulence theory}

Before we discuss the theory, let’s establish a baseline.

What observational facts do we know about what we can use to constrain theory?

1) $2/3$ law: the mean square velocity increment $\langle \delta u^2 \rangle$ between two points separated by $\ell$ scales as $\langle \delta u^2 \rangle \sim \ell^{2/3}$

2) Finite energy dissipation: the energy dissipation is always positive and finite.
In 1942, Richardson conjectured that the energy transfer is local in space to the viscous scale

but this conjecture alone does not reproduce the observable aspects of turbulence above.

So, in 1941, Kolmogorov proposed the first theory that did explain (1) & (2) above. To do so, he assumed the following:

1) Universality: The turbulence inertial range is independent of the particular forcing (and dissipation)
2) Locality of interactions
3) Homogeneity: No special points \( \Rightarrow \) no intermittency
4) Isotropy: No special directions
5) Scale invariance: No special scales \( \Rightarrow \) constant cascade rate, \( \varepsilon \)

Scale invariance \( \Rightarrow \varepsilon = \text{const} \), but we already argued that \( \varepsilon \sim \frac{u_0^2}{\ell} \). Since \( \varepsilon = \text{const} \),

\( \varepsilon \sim \frac{5u_0^2}{\ell} \). \( \therefore \) \( \delta u_L \sim (\varepsilon \ell)^{1/3} \Rightarrow \delta u_L^2 \sim \varepsilon^{2/3} \ell^{2/3} \)

and we have the \( 2/3 \) law!

\( \delta u_L \sim (\varepsilon \ell)^{1/3} \) is referred to as the Kolmogorov-Obukhov law.
• Energy spectra

In general, \( E = k_x, k_y, k_z \), so \( E_{(3)}(E) \) is the 3D energy spectrum. The total energy is then \( E = \int dE E_{(3)}(E) \). If the energy is isotropic in \( k \) space (\( k \leq 3 \)), then we can use spherical coordinates

\[
E = \iiint d\Omega \propto k^2 \sin \theta \sin^2 \theta E_{(3)}(E) = \int dk \propto k^2 E_{(3)}(k) = \int dk E_{(1)}(k)
\]

where \( E_{(1)}(k) \) is the 1D energy spectrum.

\[
E = 8\pi k^2 \Rightarrow 8\pi k^2 \Rightarrow 8\pi k^{2/3} \Rightarrow 8\pi k^{2/3} \Rightarrow \int dk E_{(3)}(k) = \int dk E_{(1)}(k)
\]

\[
E_{(1)} \propto k^{5/3} \quad \text{and} \quad E_{(3)} \propto k^{11/3}
\]

Normally use the 1D spectrum.

What about scales \( k < k_0 \) and \( k > k harmonics? \)

1) \( k \approx k_0 \) \( E = \rho_0^2 = \text{const} \Rightarrow E_{(1)} \propto k^{-1} \)

2) \( k < k_0 \) \( E \sim \rho \nabla \rho \) \( u \nabla^2 u \sim \eta \frac{\partial u}{\partial x} \)

\[
\Rightarrow \int dk E_{(1)} \sim \left( \frac{L}{V} \right)^{1/2} \Rightarrow E_{(1)} \sim k^{-3}
\]

Inertial range is the range of scales unaffected by driving or dissipation. The physics is assumed to be self-similar (fractal) here.

Note that \( E \propto l^{2/3} \) dominated by large scales and gradients \( \nabla u \) \( \propto l^{-1/2} \) dominated by small scales \( \Rightarrow \text{viscous cutoff}. \)

Also, the cascade time \( t_c = \sqrt{\frac{\rho}{\nu}} \propto l^{2/3} \) decreases with scale.
Verifying that the cascade is local

Consider motions at scales \( l_1 \) and \( l_2 \).

1) Can large scale motion shear apart small scales before the cascade?

Shearing time:

\[
\tau_s \sim \frac{l_1}{\delta u_x}
\]

Cascade time for \( l_2 \):

\[
\tau_c \sim \frac{l_2}{\delta u_x} \sim \frac{l_2}{l_1} \left[ \frac{\delta u_x}{\delta u_x} \right]^{1/3} \sim \left( \frac{l_1}{l_2} \right)^{2/3} \gg 1 \Rightarrow \text{shearing by large scales not important.}
\]

2) Can small scale eddies diffuse the large eddies before they cascade?

Diffusion coefficient due to eddies of size \( l_2 \):

\[
D = \frac{l_2^2}{\tau_c} = \frac{l_2}{\delta u_x}
\]

Time to diffuse distance \( l_1 \) is:

\[
\tau_D \sim \frac{l_1^2}{D} \sim \frac{l_1}{l_2} \left[ \frac{\delta u_x}{\delta u_x} \right]^{1/3} = \left( \frac{l_1}{l_2} \right)^{4/3} \gg 1 \Rightarrow \text{diffusion is unimportant}
\]
Additional Reading


2) Landau B Lifshite "Fluid Mechanics" 1987
   Chapter 3

3) Kolmogorov 1941. Translation by Leon 1951
   "The local structure of turbulence in incompressible
   viscous fluid for very large Reynolds numbers"

4) P.A. Davidson "Turbulence: An Introduction" 2004