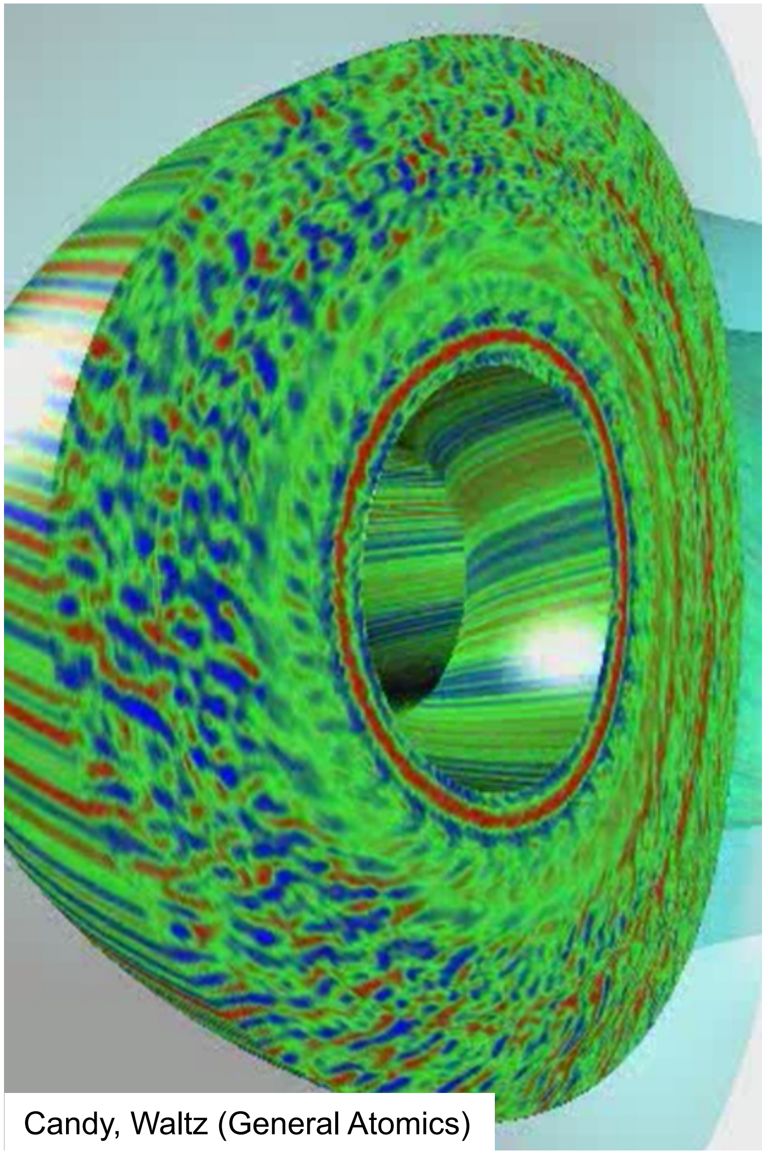
GSS Turbulence Lecture #2
Introduction to turbulence in tokamaks


Candy, Waltz (General Atomics)

Greg Hammett
w3.pppl.gov/~hammett

Graduate Summer School 2018
Princeton Plasma Physics Lab (PPPL)
Aug. 13-17, 2018.

I. **Brief Fusion & Tokamak Intro**

II. **Physical picture of main mechanism driving turbulence in tokamaks:**

   bad-curvature / effective-gravity instability in curved magnetic fields

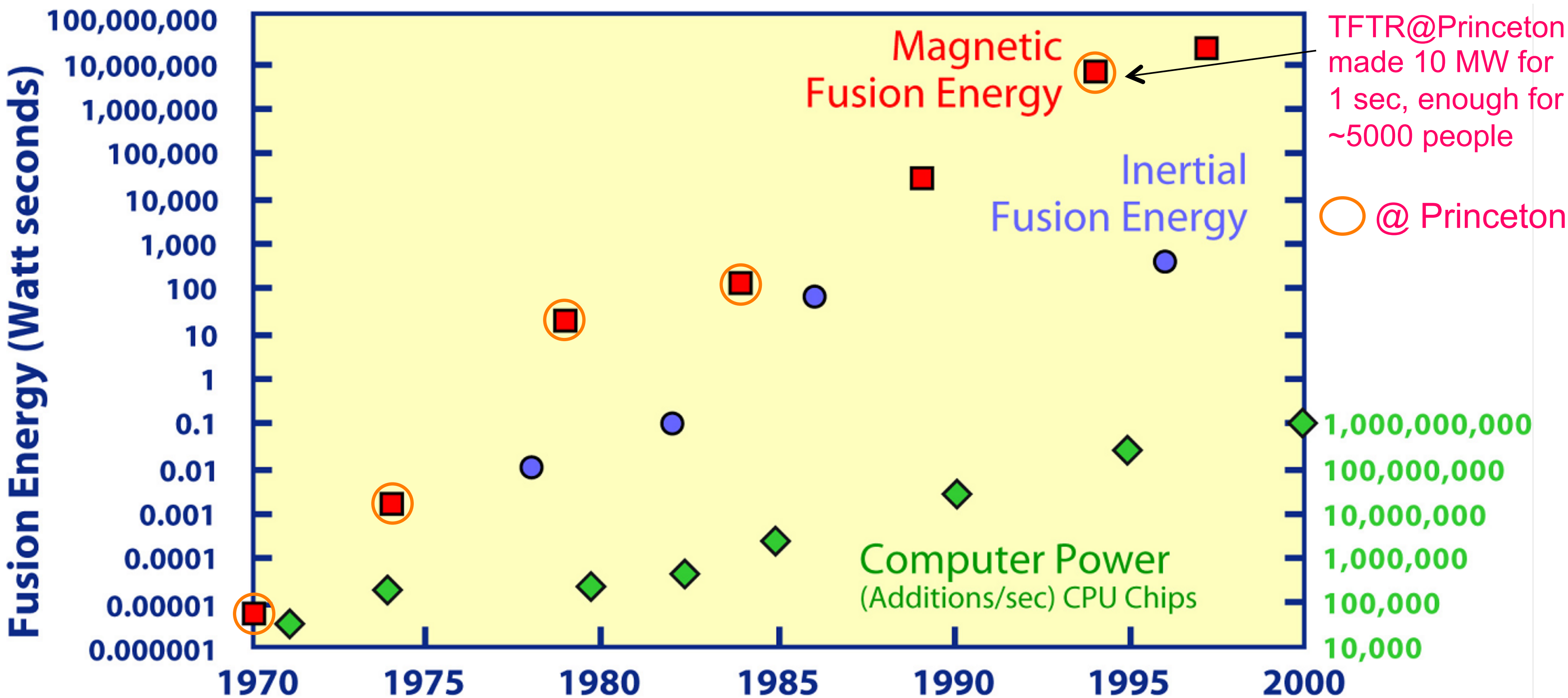
   analogies with: inverted-pendulum & Rayleigh-Taylor instabilities.

My perspective on fusion:

- Fusion energy is hard and it will take time to develop, but
  - it's an important problem,
  - we've been making progress, and
  - there are interesting ideas to pursue that could bring down the cost significantly and make it more practical

- For more on my perspective on fusion, and an introduction to the science, see video lecture at https://suli.pppl.gov/2015/course/

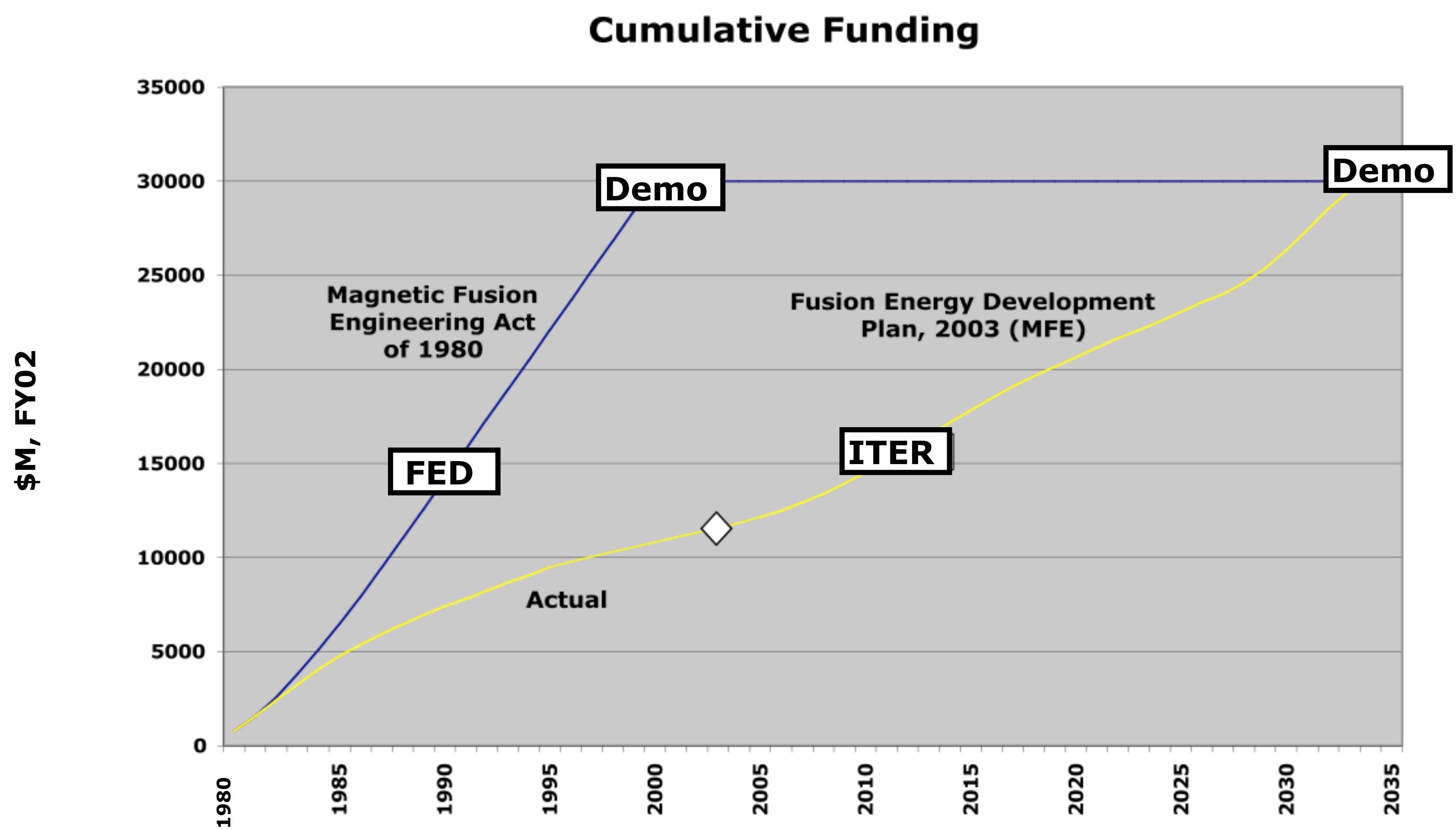
Progress in Fusion Energy has Outpaced Computer Speed



ITER goal to produce 200,000 MJ/pulse (~300 MW), 10^7 MJ/day of fusion heat).
NIF goal to produce 20 MJ/pulse (and /day) of fusion heat.

Some of the progress in computer speed can be attributed to plasma science.

Fusion Research Has Never Received Budget Needed To Fully Developed It



Plot from R.J. Goldston in 2003

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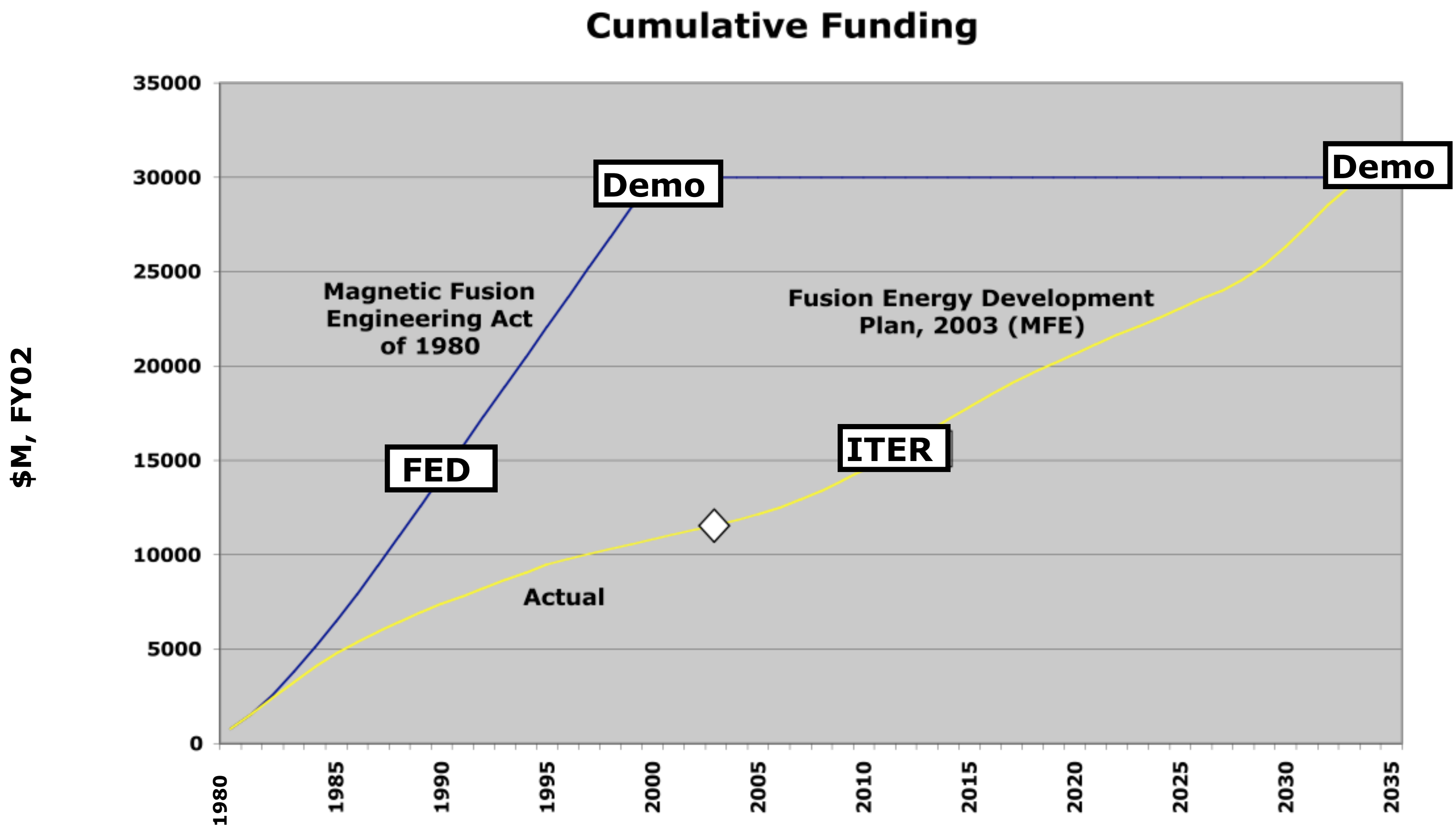
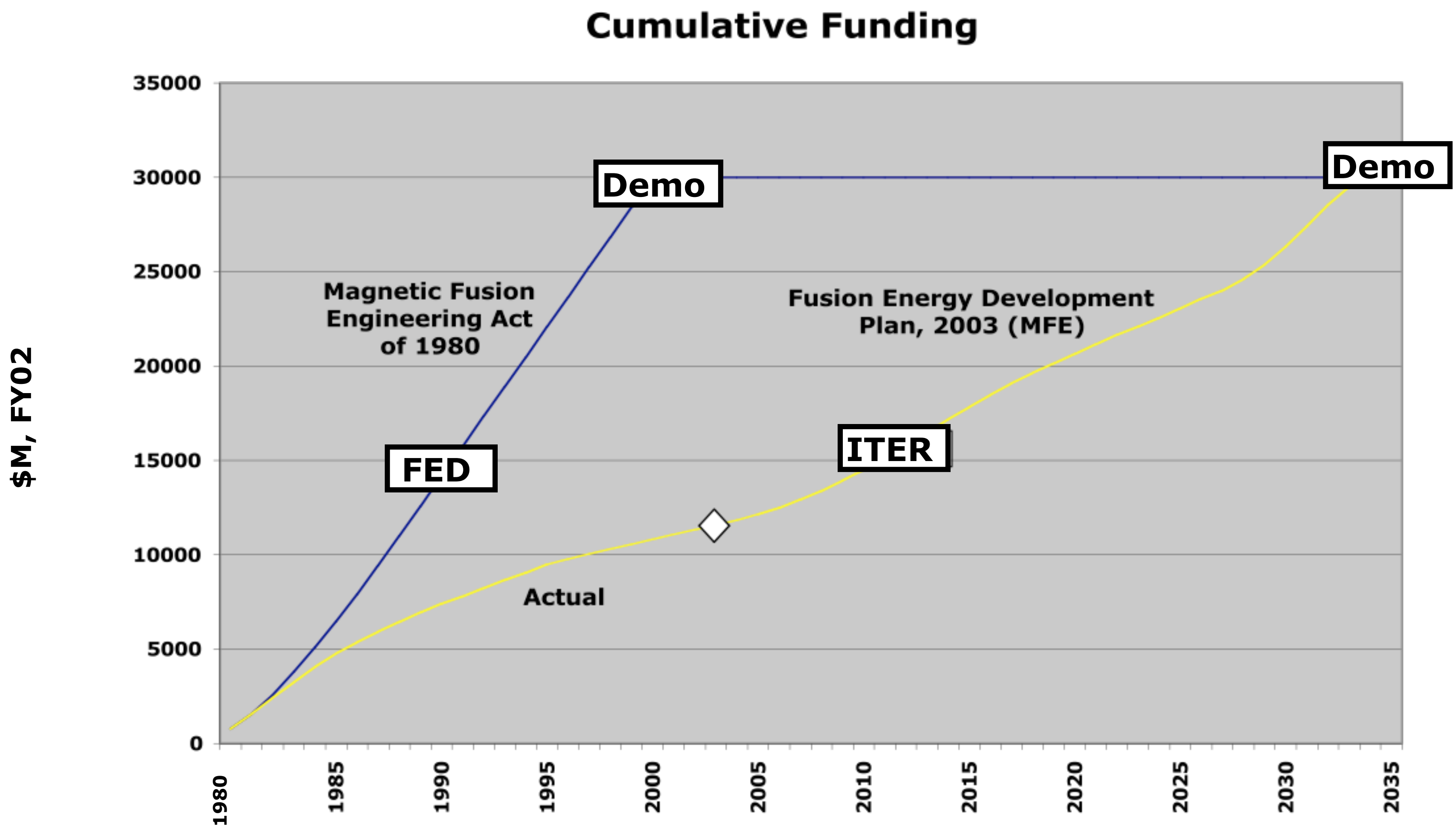


Einstein: Time is relative,

Plot from R.J. Goldston in 2003

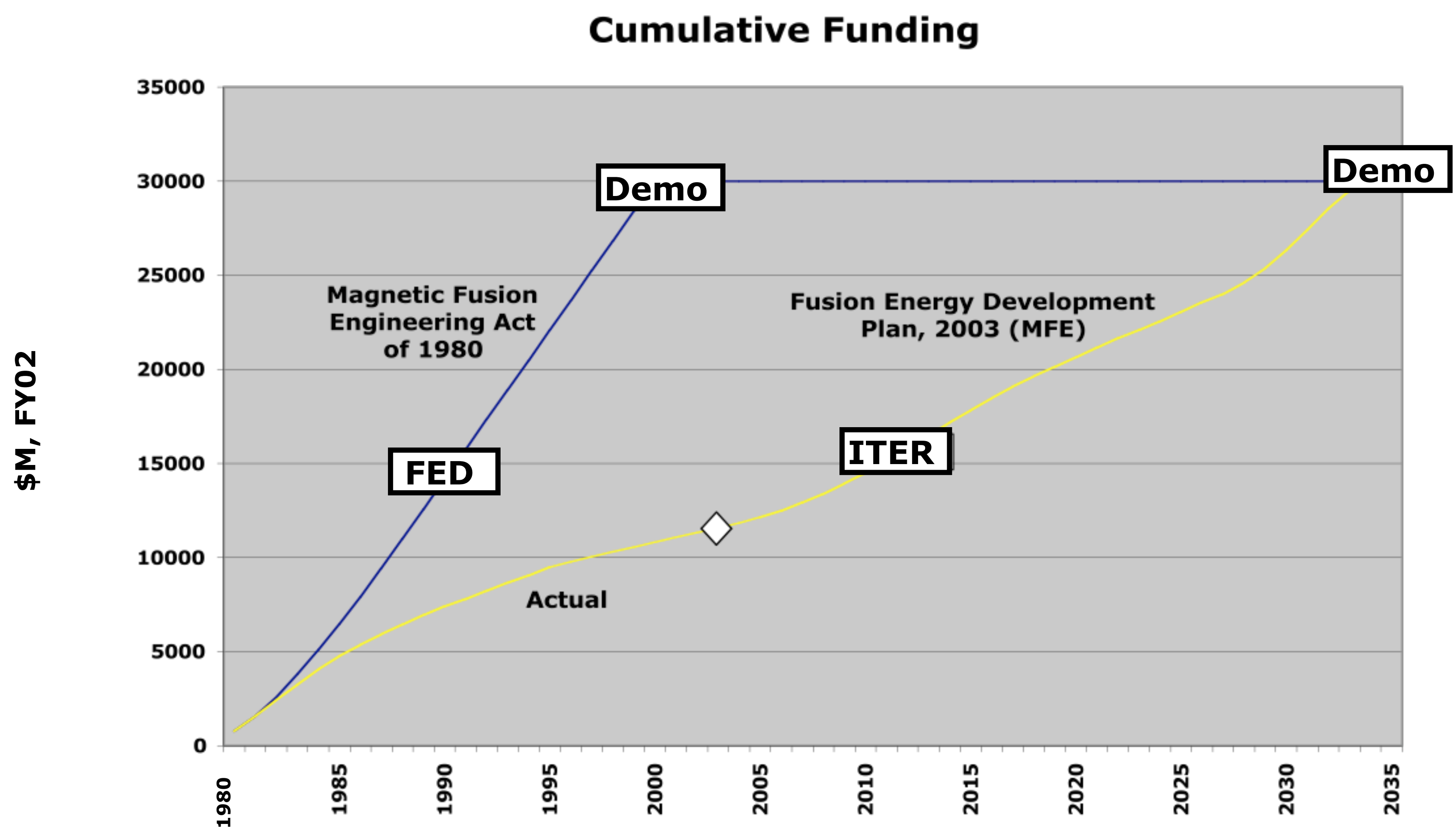
Fusion Research Has Never Received Budget Needed To Fully Developed It



Einstein: Time is relative,
Measure time in $$

Plot from R.J. Goldston in 2003

Fusion Research Has Never Received Budget Needed To Fully Developed It



The fusion program should do the best it can with the funding available to learn about fusion, find ways to improve it and bring down its cost. Aim to provide the scientific basis for a larger funding initiative someday to fully develop it.

~$80B total development cost is tiny compared to >$100 Trillion energy needs of 21st century & potential costs of global warming. Still 67:1 payoff after discounting 50+ years if fusion is just 10% cheaper than best environmentally acceptable alternative. Goldston IAEA 2006 http://www-naweb.iaea.org/napc/physics/fec/fec2006/html/node132.htm

Plot from R.J. Goldston in 2003

Need to aggressively pursue a portfolio of alternative energies in the near term (10-30 years)

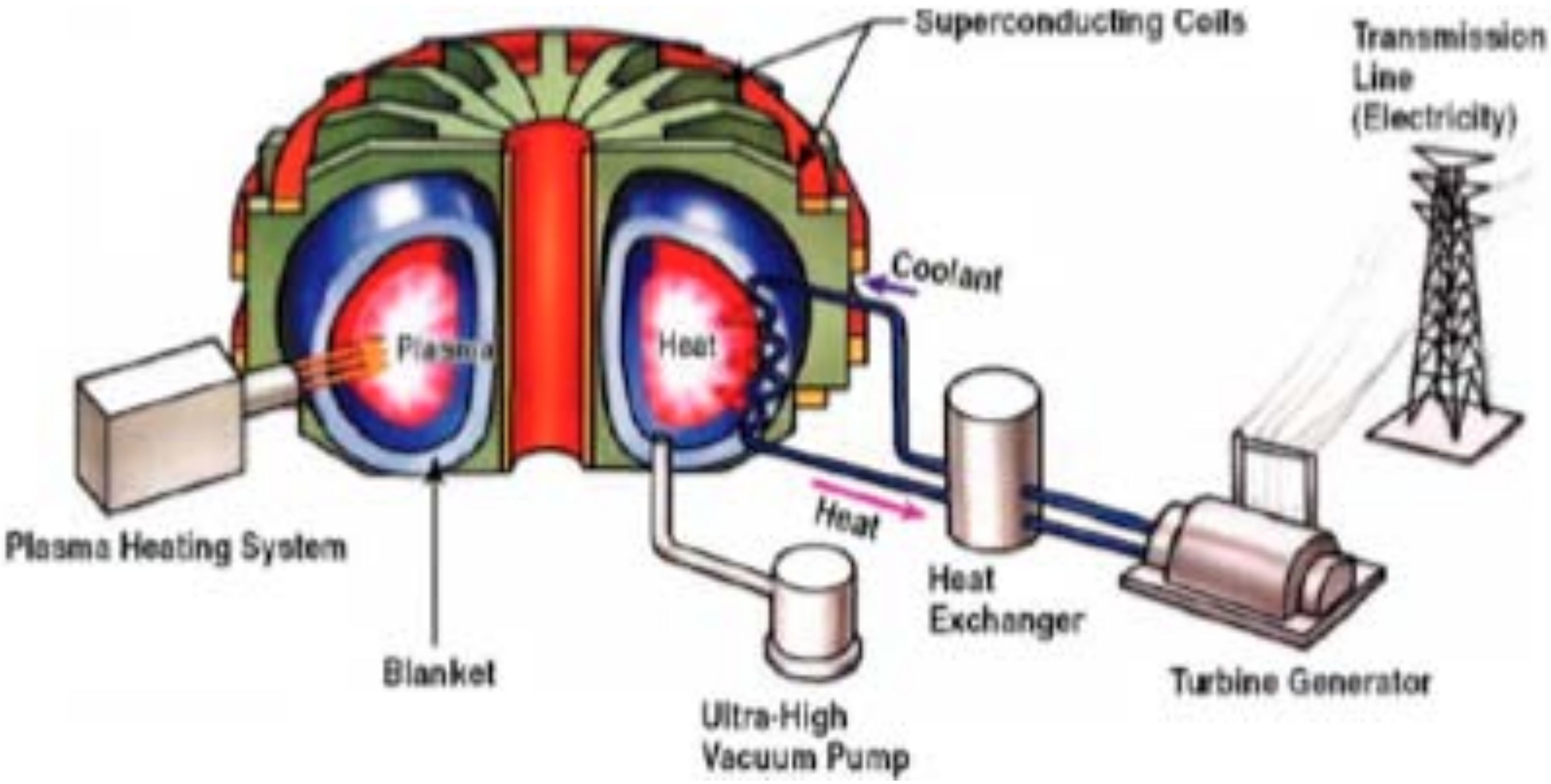
Needed to deal with global warming, energy independence, & economic growth

- improved building & transportation efficiency
- plug-in hybrid, CNG vehicles
- wind power
- concentrated solar
- photovoltaic
- storage (hourly, daily, monthly, seasonal)
- clean coal with CO_2 sequestration
- synfuels+biomass with CO_2 sequestration
- fission nuclear power plants
- …

However, there are uncertainties about all of these energy sources: cost, quantity, intermittency, storage, side-effects. How much CO_2 can be stored underground long term, and at what cost? Energy demand expected to > triple throughout this century as poorer countries continue to develop.

Because of uncertainties, particularly on the longer time scale (>50 years), still need to explore fusion.

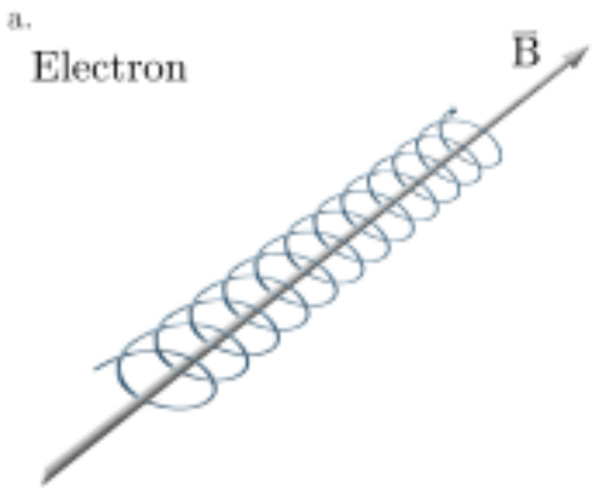
A fusion power plant

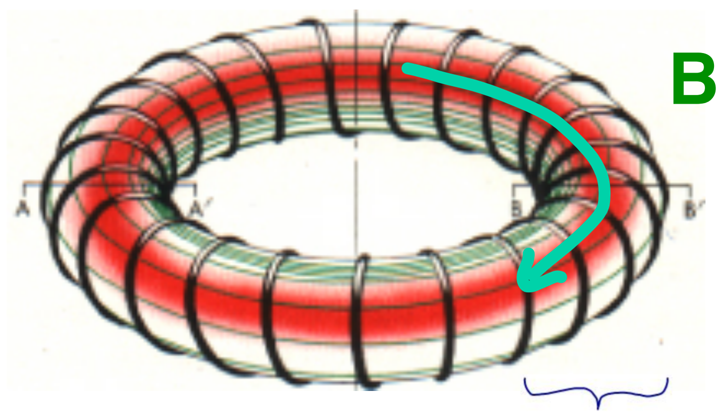


not to scale

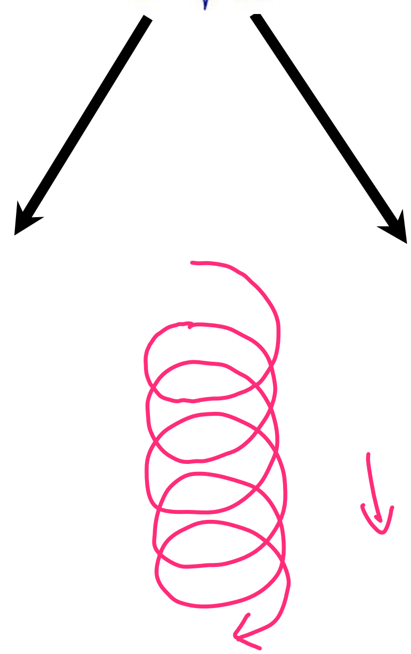
A Crash Course in Magnetic Confinement (in a few slides)

Particles have helical orbits in B field, not confined along B. Try to fix by wrapping B into a torus.





$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}$$



↓ ions drift down

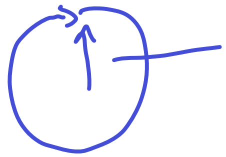
but now B ~ 1/R, so
particles will drift out:

worse than this: ions drift down & electrons drift up -->
ExB drift drives particles outward before 1 transit around torus

$$m \frac{d\underset{\sim}{v}}{dt} = q\left(\underset{\sim}{\bar{E}} + \frac{\underset{\sim}{v} \times \underset{\sim}{\bar{B}}}{c}\right) + \text{Collisions}$$

$$\rho_i = \frac{v_\perp}{\Omega_{ci}}$$

$\underset{\sim}{B}$ $\odot$



$\rho_i \sim 3$ mm for 10 keV Deuterons $B = 5$ T

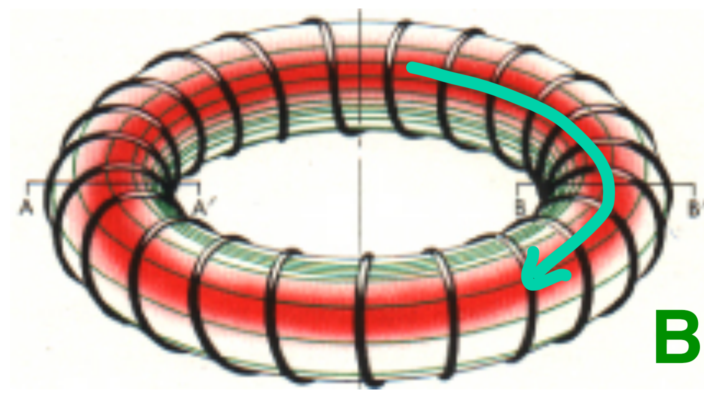
gyrofrequency ~ 40 MHz

$$\Omega_{ci} = 2\pi f_{ci} = \frac{e_i B}{m_i c}$$

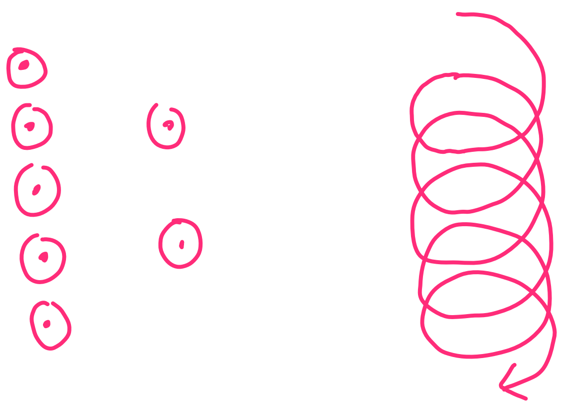
$\lambda_{mfp} \sim 10$ km for $n \sim 10^{20}/m^3$

Particles not confined along $\underset{\sim}{B}$

so wrap into a torus:



Problem: Non-uniform $|\underset{\sim}{B}| \Rightarrow$ particle drift



$\downarrow$ ions drift down

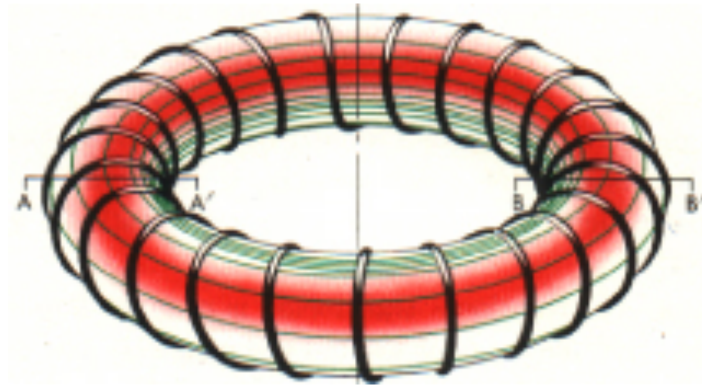
$$\vec{B} = B\hat{b}$$

$$\vec{v}_d \approx \frac{v_\parallel^2 + v_\perp^2/2}{\Omega_{ci} B} \hat{b} \times \nabla B$$

$$\sim \frac{v_t^2}{\Omega_{ci} R} \sim v_t \frac{\rho}{R} \sim v_t \frac{3\,mm}{6\,m}$$

Even worse: Charge imbalance

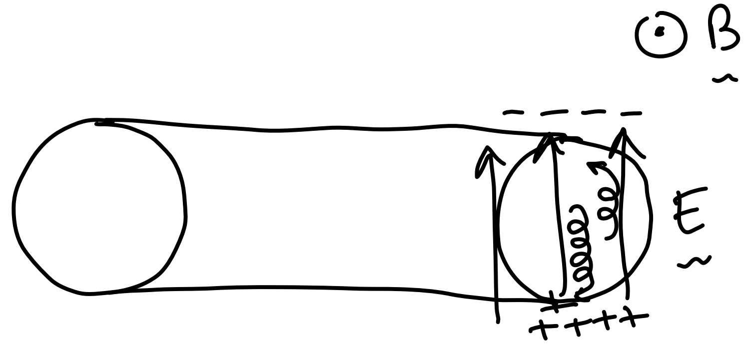
$\Rightarrow$ Instability



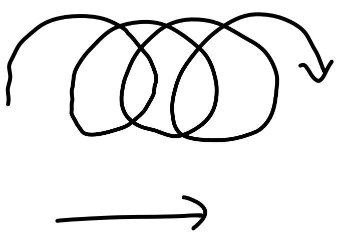
$\leftarrow \nabla B \quad \odot \underset{\sim}{B}$

ions drift down

$\uparrow$ electrons drift up

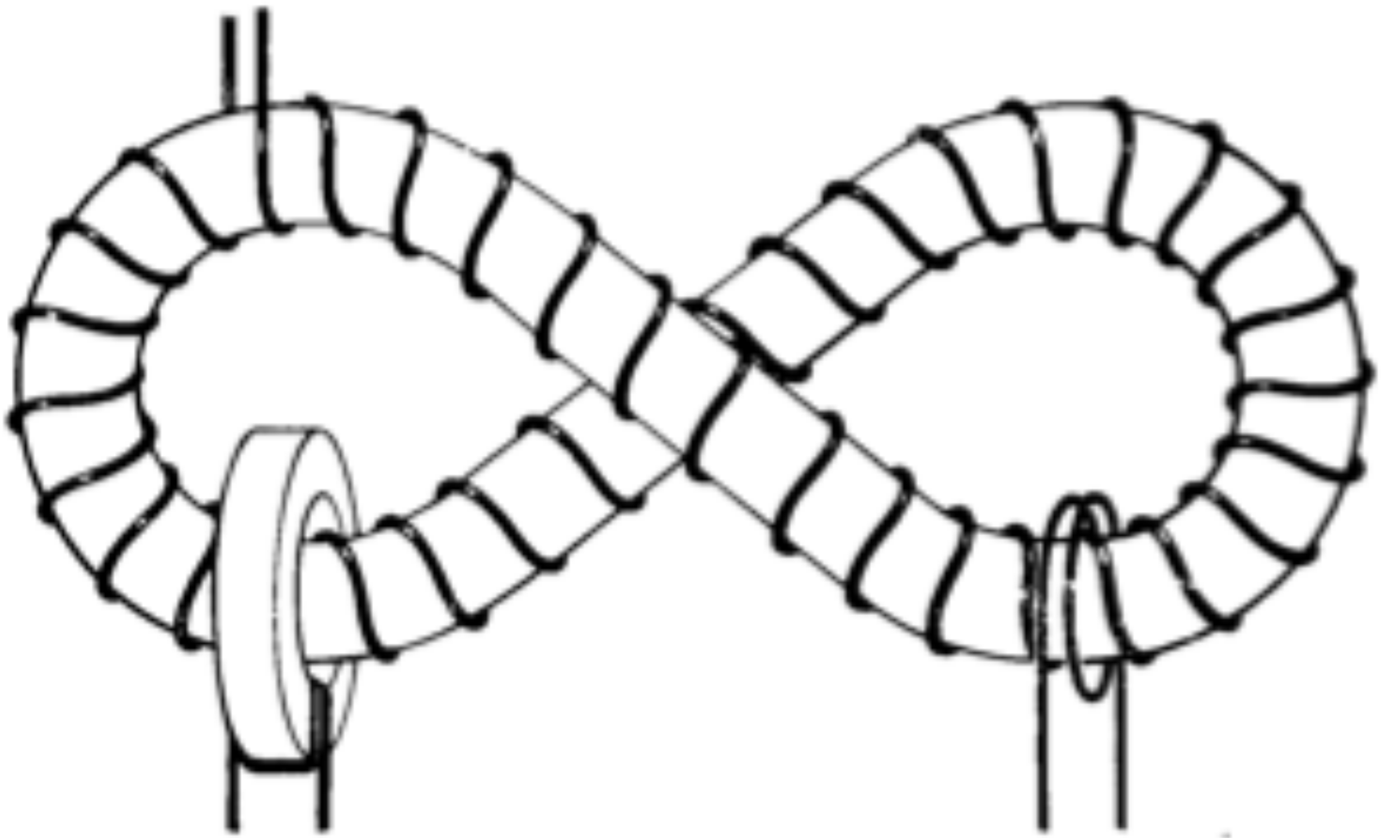


Particle motion in $\underset{\sim}{E}$ & $\underset{\sim}{B}$:



$$\underset{\sim}{v}_{drift} = \frac{c}{B^2}\, \underset{\sim}{E} \times \underset{\sim}{B}$$

Spitzer's stellarator solution: twist torus into figure-8 to cancel drifts and confine particles.



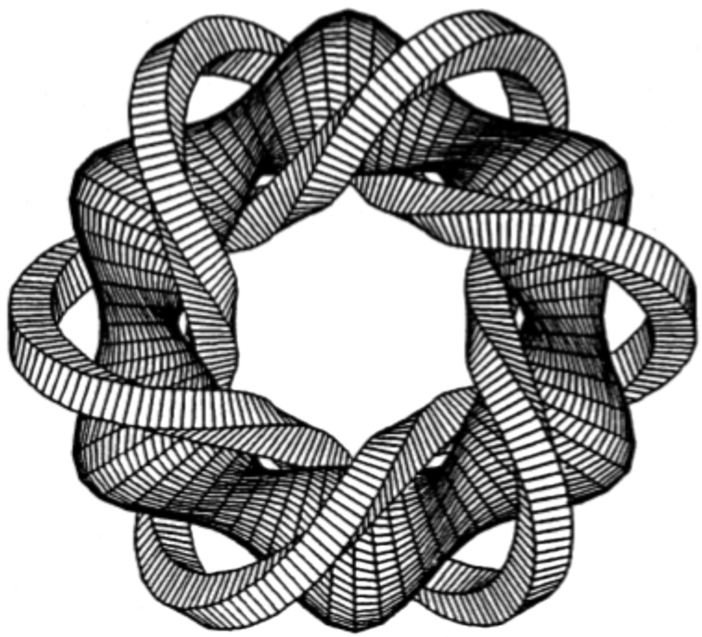
ions drift out of the page on one side of figure 8, but drift into page on other side.

(Also, electrons can flow along field lines to shield charge buildup.)

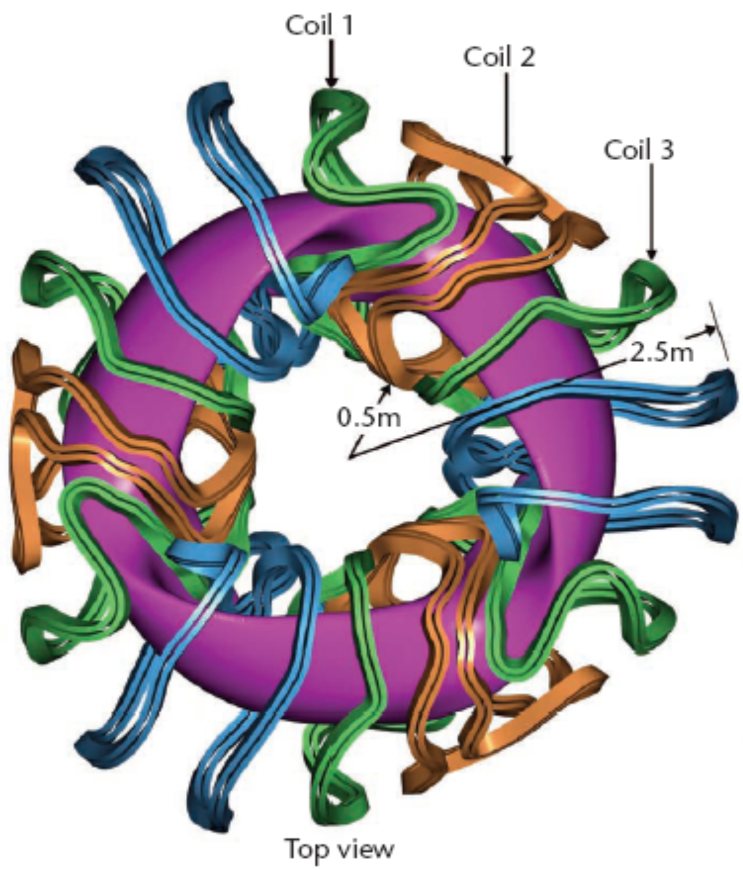
R. A. Ellis, Jr,, Princeton Alumni Weekly, Sept. 19, 1958

Modern stellarators

Spitzer et al. later realized that particles can be confined by a net poloidal twist in the magnetic field produced by helical coils. (First realization of the "Berry Phase"*.) Eventually evolved into modern stellarator designs with modular, unlinked coils.



JF Lyon et al., 1997 http://aries.ucsd.edu/LIB/REPORT/SPPS/FINAL/chap2.pdf



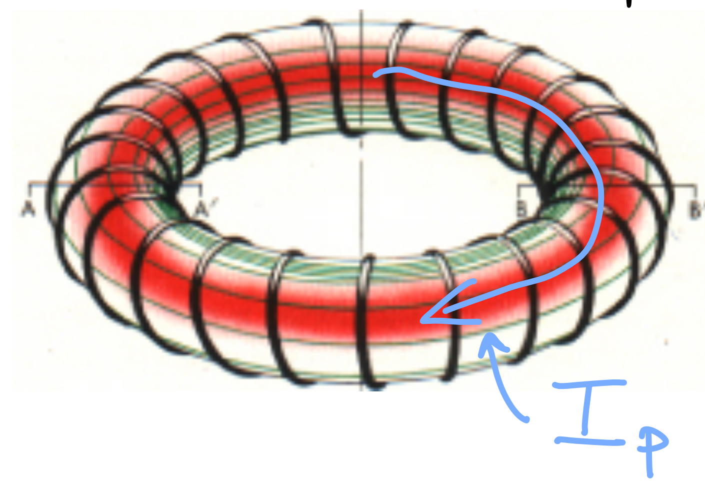
Princeton Quasar (Quasi-axisymmetric Stellarator)

* pointed out in Bhattacharjee, 1992

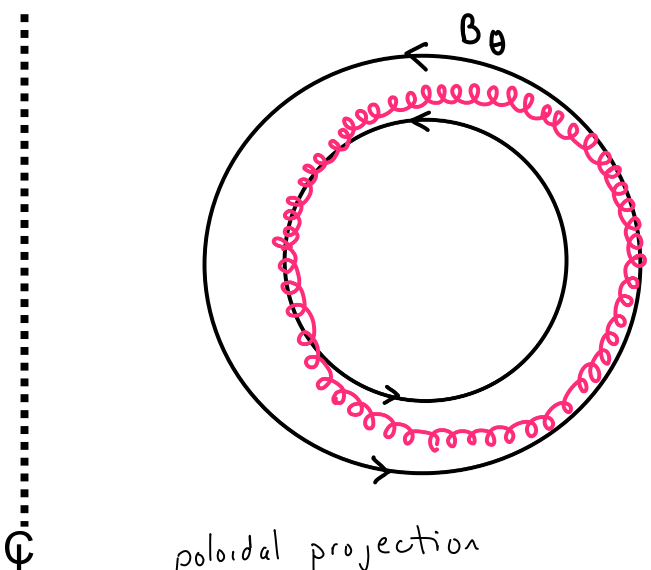
Cure problems by twisting the $\underset{\sim}{B}$ field

The Tokamak approach, invented by Sakharov (later the famous Russian dissident) and Tamm.

induce a current in plasma:

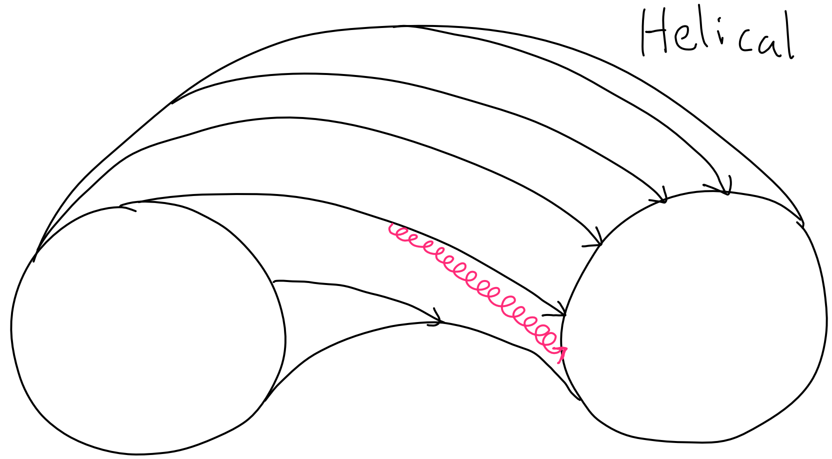


Ion motion along twisting $\underset{\sim}{B}$ field + downward drift:



poloidal projection

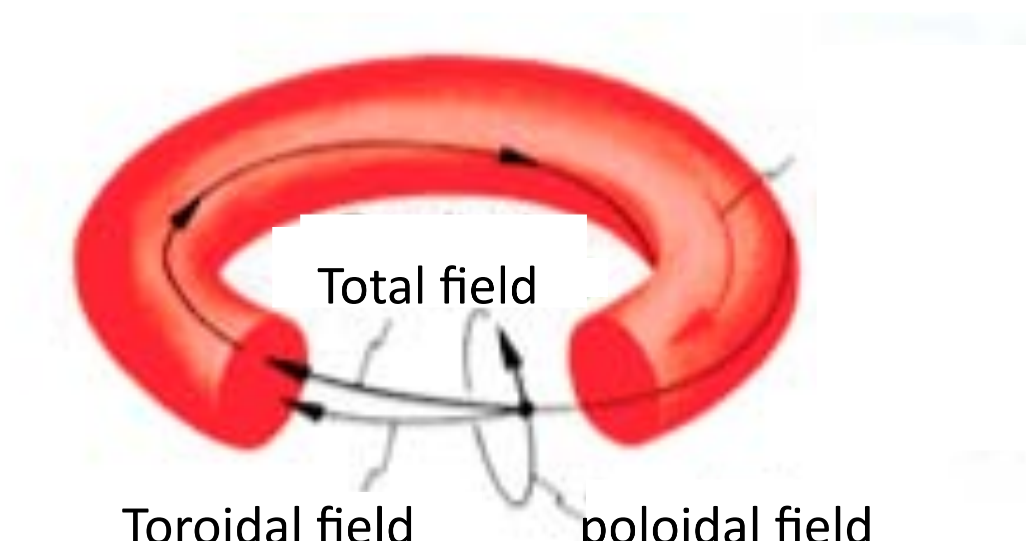
Helical twist in $\vec{B}$



Perspective view

The Tokamak

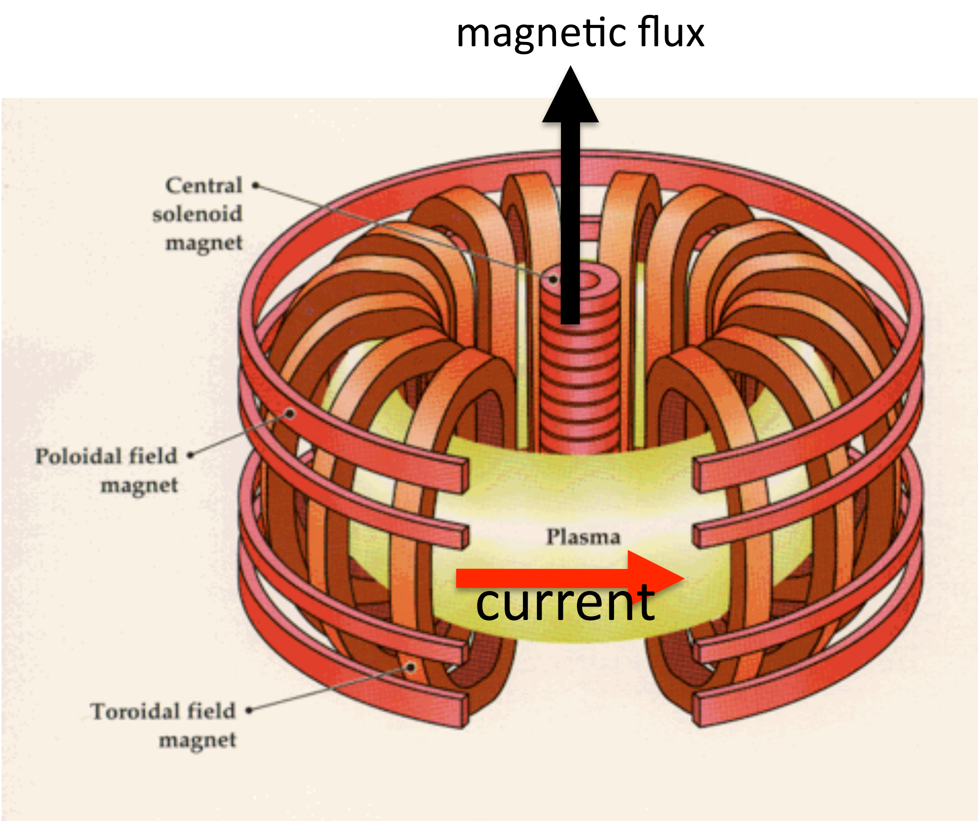
Magnetic field is helical



stability equilibrium

B is axisymmetric (2D), toroidal angle is ignorable, $\partial B / \partial \varphi = 0$

The tokamak



Magnets → $B_{toroidal}$

Plasma current → $B_{poloidal}$

Time-varying Magnetic flux → Plasma current

Improving Confinement Can Significantly ↓ Size & Construction Cost of Fusion Reactor

Well known that improving confinement & β can lower Cost of Electricity / kWh, at fixed power output.

Even stronger effect if consider smaller power: better confinement allows significantly smaller size/cost at same fusion gain Q ($nT\tau_E$).

Standard H-mode empirical scaling:

$$\tau_E \sim H\, I_p^{0.93}\, P^{-0.69}\, B^{0.15}\, R^{1.97} \ldots$$

($P = 3VnT/\tau_E$ & assume fixed $nT\tau_{E,}\ q_{95},\ \beta_N,\ n/n_{Greenwald}$):

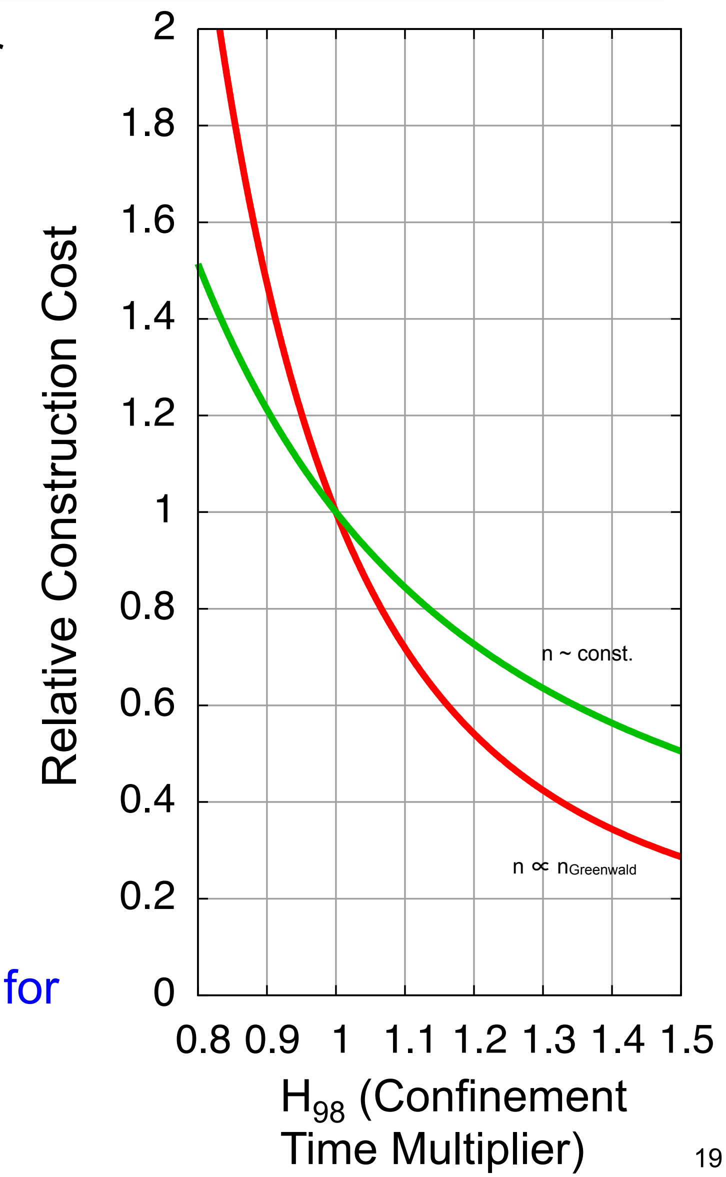
$$\$ \sim R^2 \sim 1 / (H^{4.8} B^{3.4})$$

ITER std $H=1$, steady-state $H{\sim}1.5$

ARIES-AT $H{\sim}1.5$

MIT ARC $H_{89}/2 \sim 1.4$

Need comprehensive simulations to make case for extrapolating improved H to reactor scales.

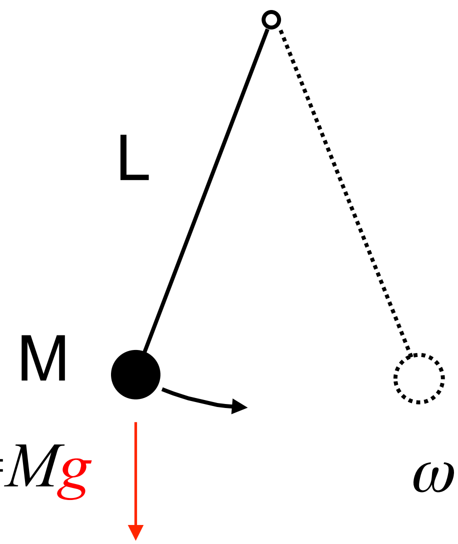


(Plots assumes cost ∝ R^2 roughly. Includes constraint on B @ magnet with ARIES-AT 1.16 m blanket/shield, $a/R=0.25$, i.e. $B = B_{mag}\,(R-a-a_{BS})/R$. Neglects current drive issues.)

Part 1: Intuitive picture of the ITG instability

-- based on analogy with Inverted pendulum / Rayleigh-Taylor instability

Stable Pendulum

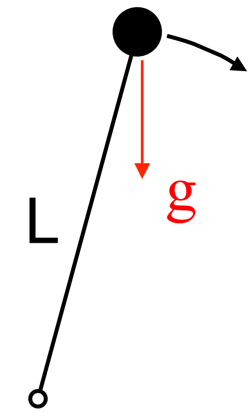


$F=Mg$

$\omega=(g/L)^{1/2}$

Unstable Inverted Pendulum

(rigid rod)

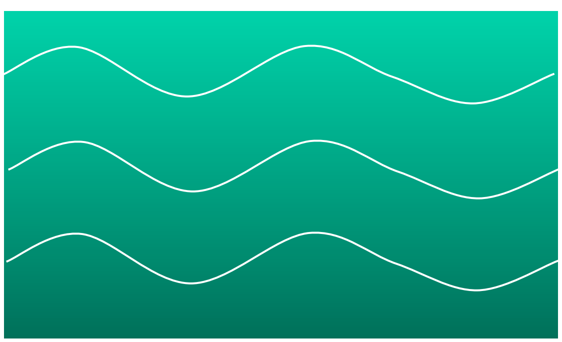


$\omega= (-g/|L|)^{1/2} = i(g/|L|)^{1/2} = i\gamma$

Instability

Density-stratified Fluid

$\rho=exp(-y/L)$

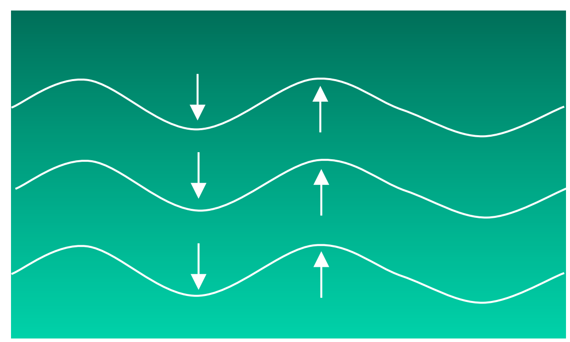


stable $\omega=(g/L)^{1/2}$

Inverted-density fluid

⇒Rayleigh-Taylor Instability

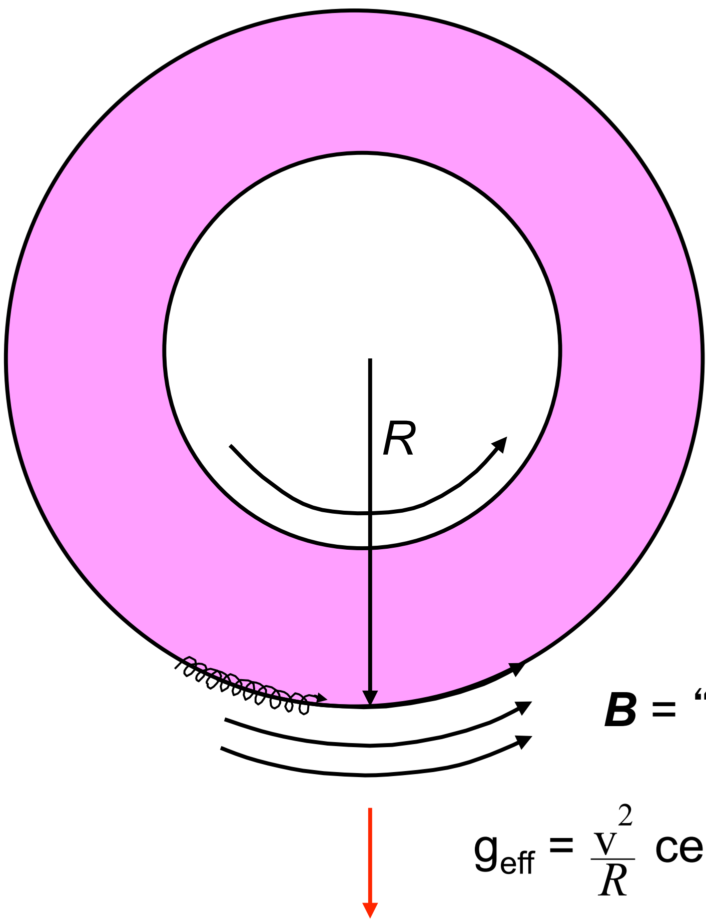
$\rho=exp(y/L)$



Max growth rate $\gamma=(g/L)^{1/2}$

"Bad Curvature" instability in plasmas
≈ Inverted Pendulum / Rayleigh-Taylor Instability

Top view of toroidal plasma:



plasma = heavy fluid

B = "light fluid"

$g_{eff} = \frac{v^2}{R}$ centrifugal force

Growth rate:

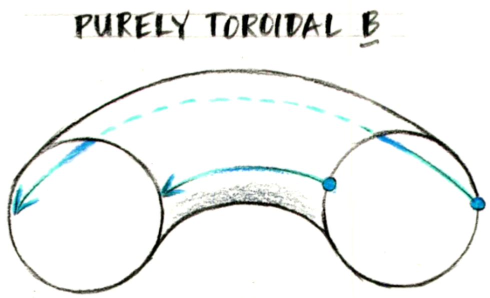
$$\gamma = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{v_t^2}{RL}} = \frac{v_t}{\sqrt{RL}}$$

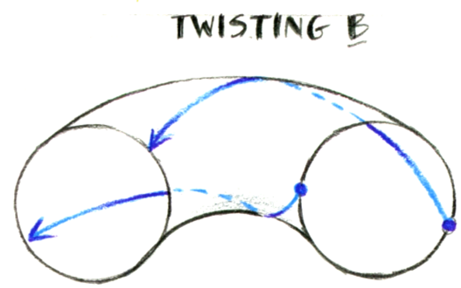
Similar instability mechanism in MHD & drift/microinstabilities

$1/L = |\nabla p|/p$ in MHD,
$\propto$ combination of ∇n & ∇T
in microinstabilities.

The Secret for Stabilizing Bad-Curvature Instabilities

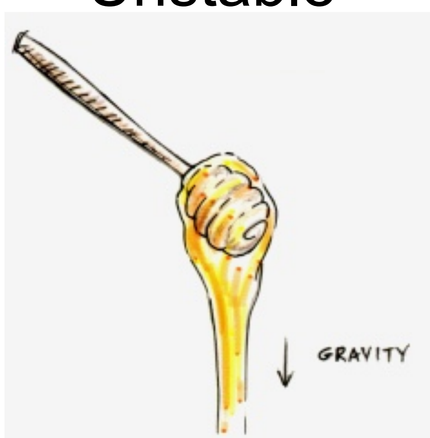
Twist in ***B*** carries plasma from bad curvature region to good curvature region:

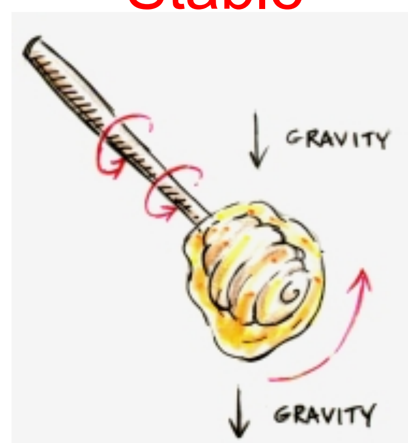




Unstable

Stable





Similar to how twirling a honey dipper can prevent honey from dripping.

Twist in B stabilizes unless

$$\boxed{\begin{array}{c} \text{growth rate} \\ \text{in bad-curvature} \\ \text{region} \end{array} > \begin{array}{c} \text{propagation from bad-curvature} \\ \text{to good curvature regions} \end{array}}$$

MHD works well to lowest order in plasmas, so RHS $\Rightarrow$

$$\frac{v_t}{\sqrt{RL}} > k_{\parallel} v_A \sim \frac{v_A}{q R}$$

Square:

$$\frac{v_t^2 \, q^2 R^2}{v_A^2 \, R L} > 1$$

$$\text{LHS} = \frac{\beta}{2} \frac{q^2 R}{L} = \frac{1}{2} q^2 R \left| \frac{\partial \beta}{\partial r} \right| = \frac{1}{2} \alpha_{MHD}$$

An aside to define some tokamak terminology (ι used in stellarator literature):

$$\iota = \text{"rotational transform" (or "twisting rate")}$$

$$q = \frac{1}{\iota} = \text{"safety factor" or "inverse rotational transform"}$$

(or "inverse twisting rate")

$q =$ # of times a field line goes around toroidally

in order to go once around poloidally

$$\mathrm{q} \approx \frac{rB_{tor}}{RB_{pol}}$$

Note: older stellarator literature (< ~ late 1990s) defined "iota bar":

$$\bar{\iota} = \iota / (2\pi) = 1 / q$$

$q \approx 1.6$ in the upper right figure 2 slides back.

While MHD works well to lowest order in plasmas, there are next-order FLR corrections that defrost the magnetic field & allow $E_{\parallel} \neq 0$ & allow the plasma to move separately from $\underset{\sim}{B}$.

Still have sound waves that can connect good & bad curvature region. Unstable if:

$$\gamma > \text{connection rate}$$

$$\frac{v_t}{\sqrt{RL}} > \frac{v_t}{qR}$$

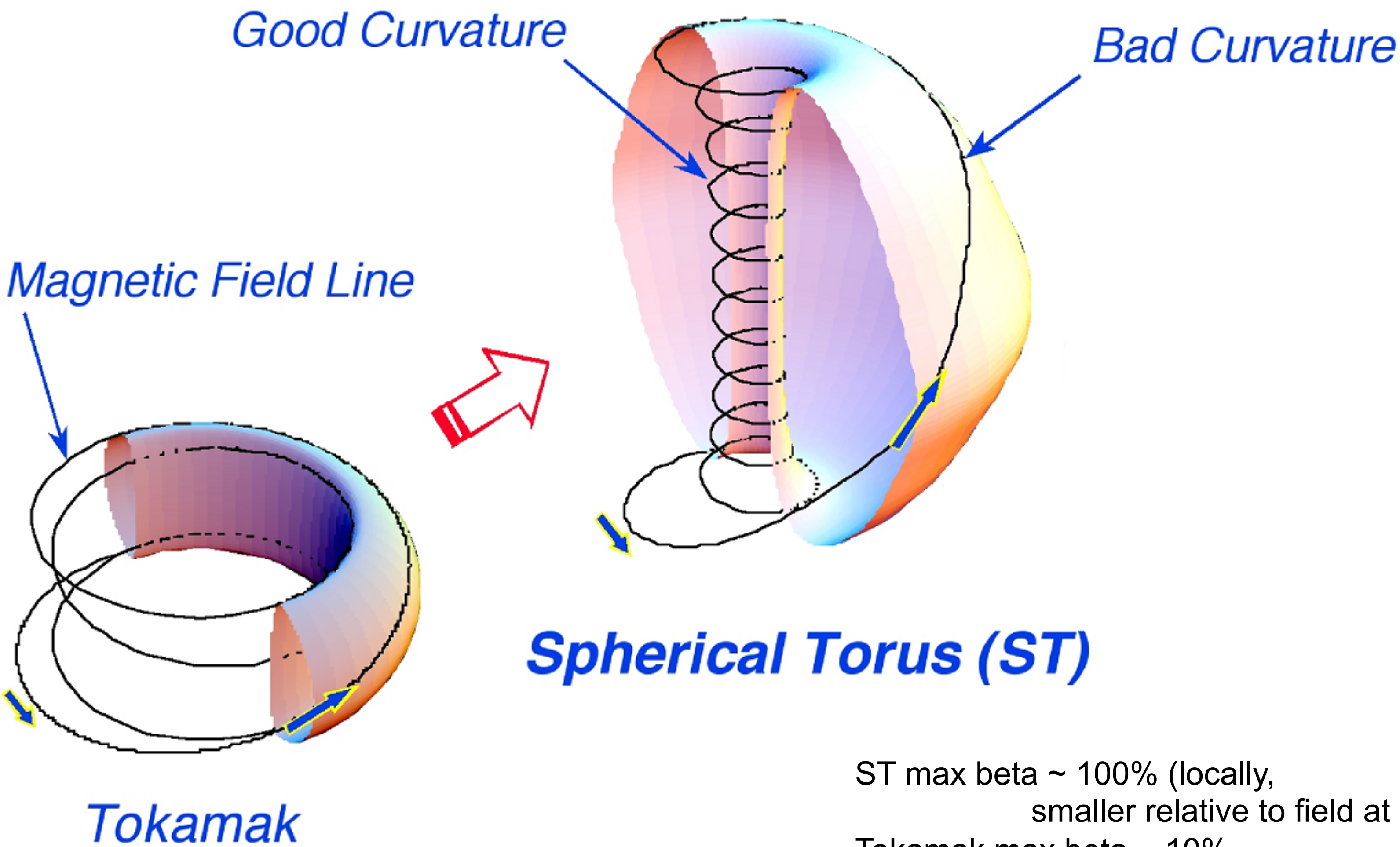
Rough, but tells us $\frac{R}{L}$ is important...

$$\boxed{\frac{R}{L} > \frac{1}{q^2}}$$

$\frac{1}{L} \sim \frac{\nabla T}{T}$

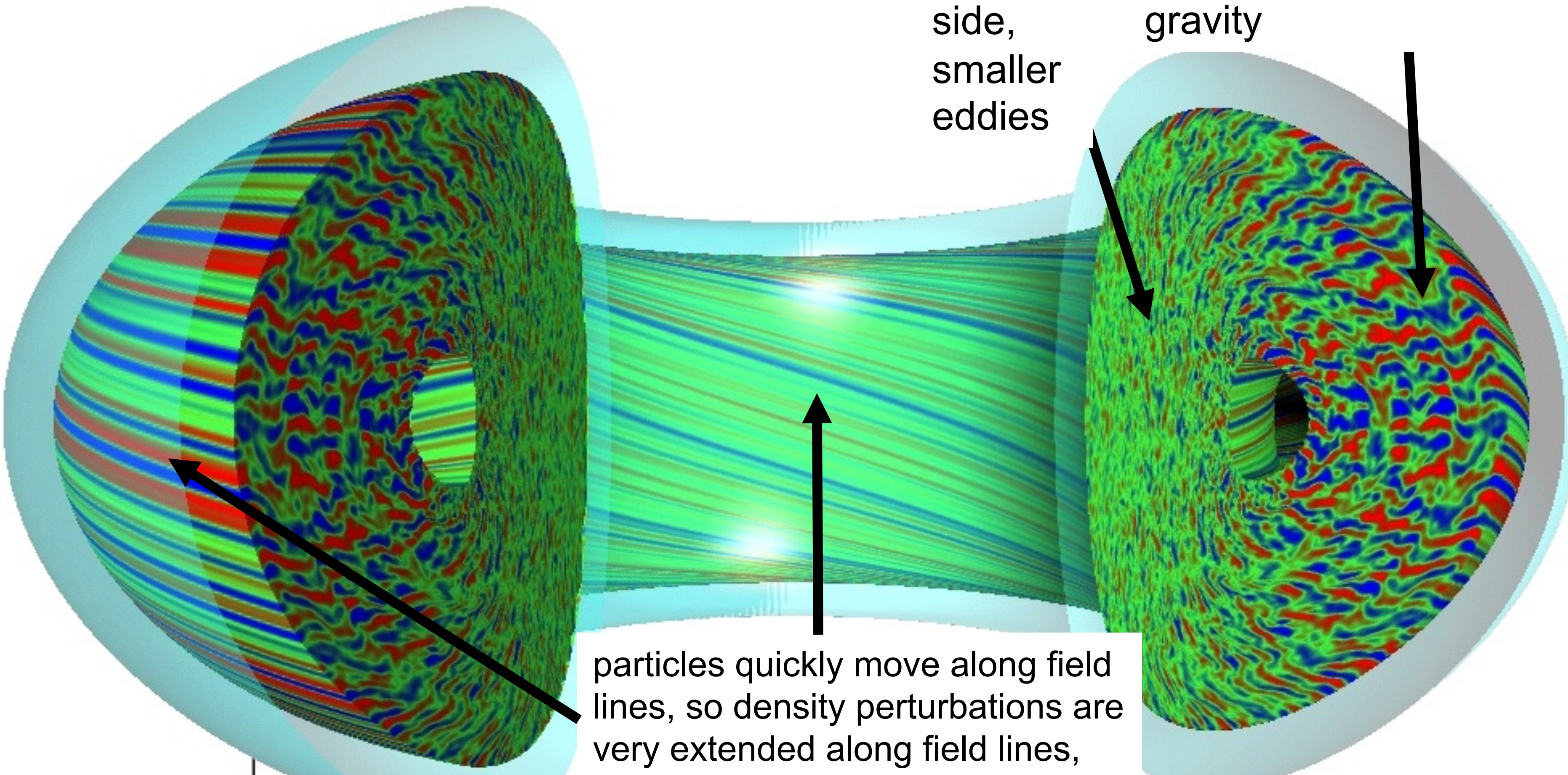
or $\sim \nabla p / p$

Spherical Torus has improved confinement and pressure limits (but less room in center for coils)



ST max beta ~ 100% (locally,
smaller relative to field at coil)
Tokamak max beta ~ 10%

These physical mechanisms can be seen in gyrokinetic simulations and movies



Note: previous and other figures show color contours of density fluctuations, not of the total density, because if one plotted contours of total density, the tiny fluctuations would not be visible:

$$n_e(\vec{x},t) = n_{e0}(r) + \delta n(\vec{x},t)$$

$$\delta n \sim 10^{-3} - 10^{-2}\, n_{e0} \quad \text{in plasma core}$$

For low-frequency fluctuations, $\omega \ll k_{||}\, v_{te}$, electrons have a Boltzmann response to lowest order along a field line:

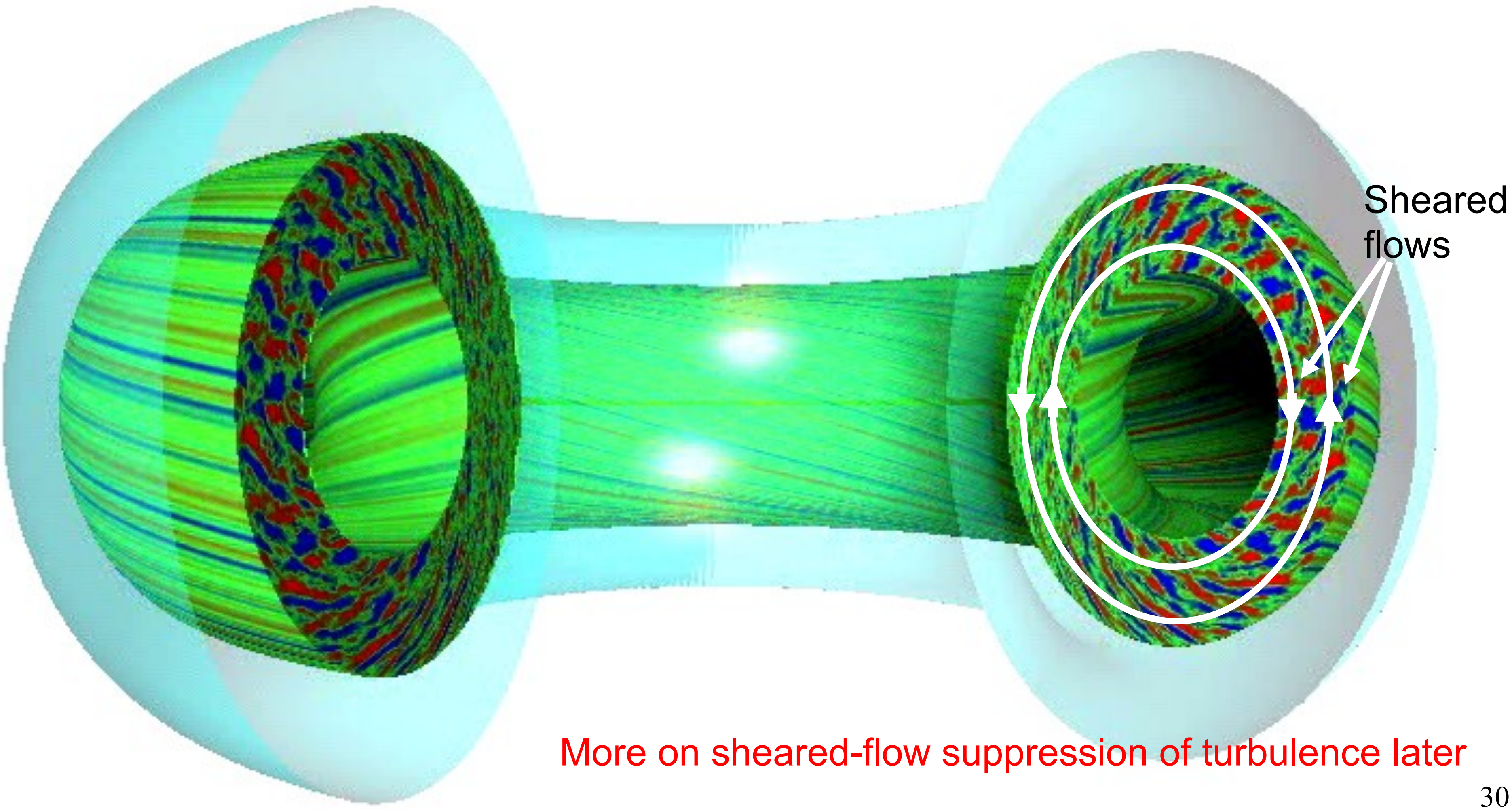
$$n_e(\vec{x},t) = C(r) e^{e\phi/T_{e0}}$$

$$\approx n_{e0}\left(1 + \frac{e(\phi - \langle\phi\rangle)}{T_{e0}}\right)$$

$$\delta n_e(\vec{x},t) \approx n_{e0}\frac{e(\phi - \langle\phi\rangle)}{T_{e0}}$$

So contours of density fluctuations are also contours for the (non-zonal) potential, and so represent stream lines for the (non-zonal) ExB drift. (Like stream lines in 2D fluid flow.) Can illustrate this with a sketch…

Movie https://fusion.gat.com/theory-wiki/images/3/35/D3d.n16.2x_0.6_fly.mpg from http://fusion.gat.com/theory/Gyromovies shows contour plots of density fluctuations in a cut-away view of a GYRO simulation (Candy & Waltz, GA). This movie illustrates the physical mechanisms described in the last few slides. It also illustrates the important effect of sheared flows in breaking up and limiting the turbulent eddies. Long-wavelength equilibrium sheared flows in this case are driven primarily by external toroidal beam injection. (The movie is made in the frame of reference rotating with the plasma in the middle of the simulation. Barberpole illusion makes the dominantly-toroidal rotation appear poloidal..) Short-wavelength, turbulent-driven zonal flows also play important role in nonlinear saturation.



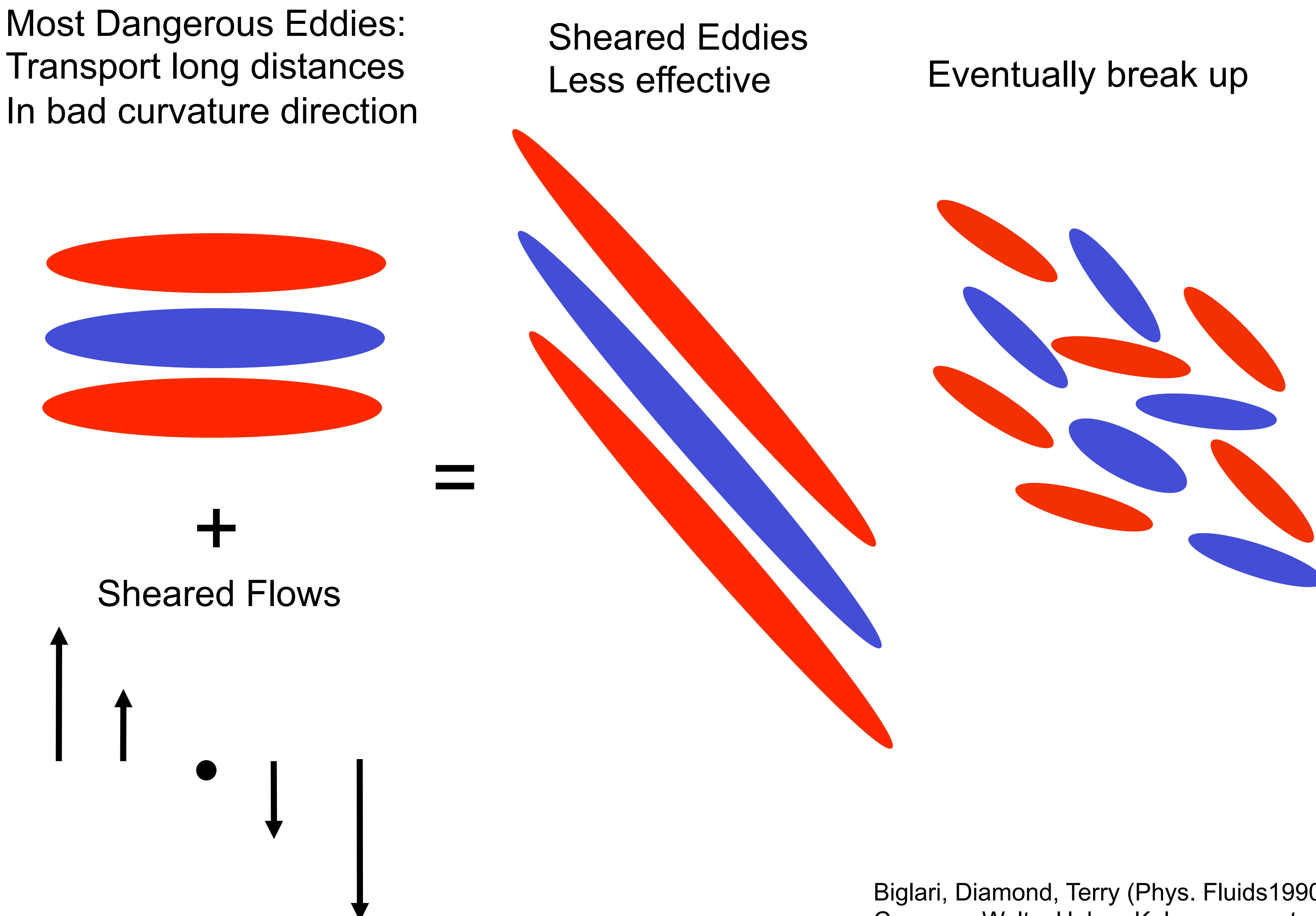
More on sheared-flow suppression of turbulence later

“Barberpole Illusion”:
https://en.wikipedia.org/wiki/Barberpole_illusion

Show supercylone.mpg movie: note

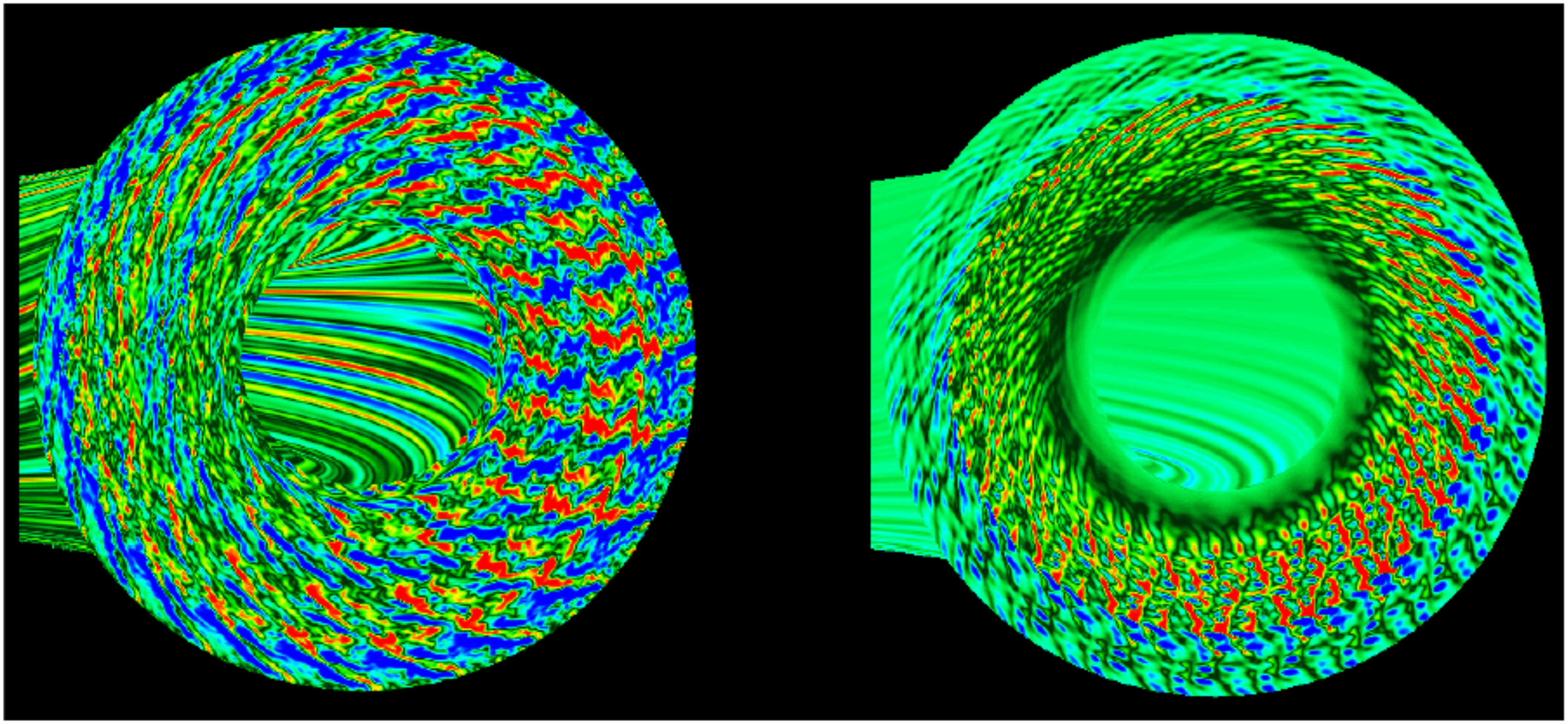
- Explain multiple wave packets as being connected along field line, decaying slowly along field line (“ballooning structure”)

Sheared flows can suppress or reduce turbulence



Biglari, Diamond, Terry (Phys. Fluids1990),
Carreras, Waltz, Hahm, Kolmogorov, et al.

Sheared ExB Flows can regulate or completely suppress turbulence (analogous to twisting honey on a fork)



Dominant nonlinear interaction between turbulent eddies and ±θ-directed zonal flows.

Additional large scale sheared zonal flow (driven by beams, neoclassical) can completely suppress turbulence

Waltz, Kerbel, Phys. Plasmas 1994 w/ Hammett, Beer, Dorland, Waltz Gyrofluid Eqs., Numerical Tokamak Project, DoE Computational Grand Challenge

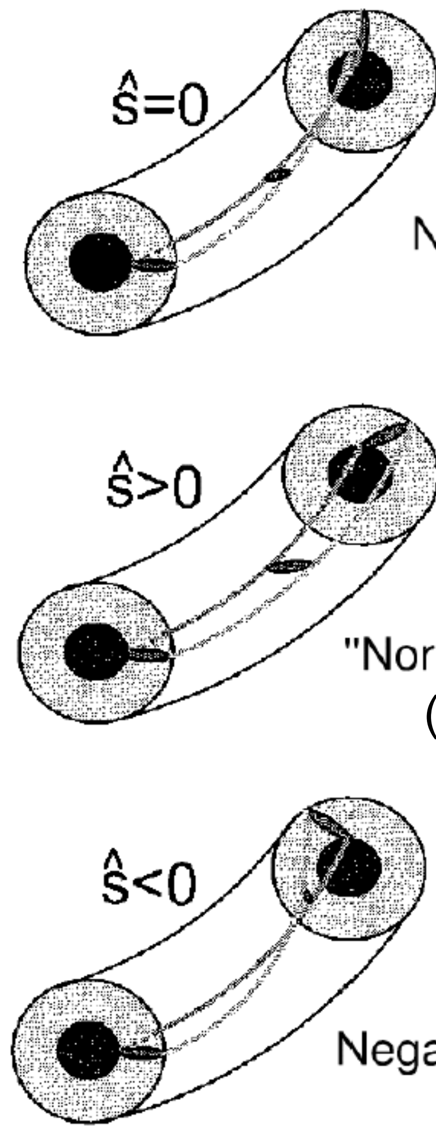
Simple picture of reducing turbulence by negative magnetic shear

Particles that produce an eddy tend to follow field lines.

Reversed magnetic shear twists eddy in a short distance to point in the “good curvature direction”.

Locally reversed magnetic shear naturally produced by squeezing magnetic fields at high plasma pressure: “Second stability” Advanced Tokamak or Spherical Torus.

Shaping the plasma (elongation and triangularity) can also change local shear



No Magnetic Shear

"Normal" Magnetic Shear
(in std tokamaks)

Advanced Tokamaks

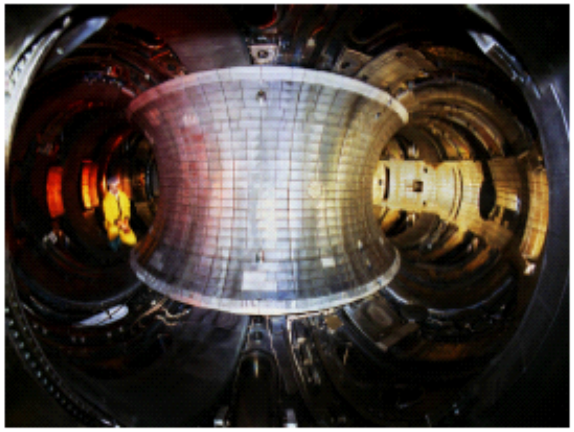
Negative Magnetic Shear
“Normal” in stellarators

Fig. from Antonsen, Drake, Guzdar et al. Phys. Plasmas 96
Kessel, Manickam, Rewoldt, Tang Phys. Rev. Lett. 94

Gyrokinetic simulation of an Enhanced Reverse Shear Plasma with an Internal Transport barrier

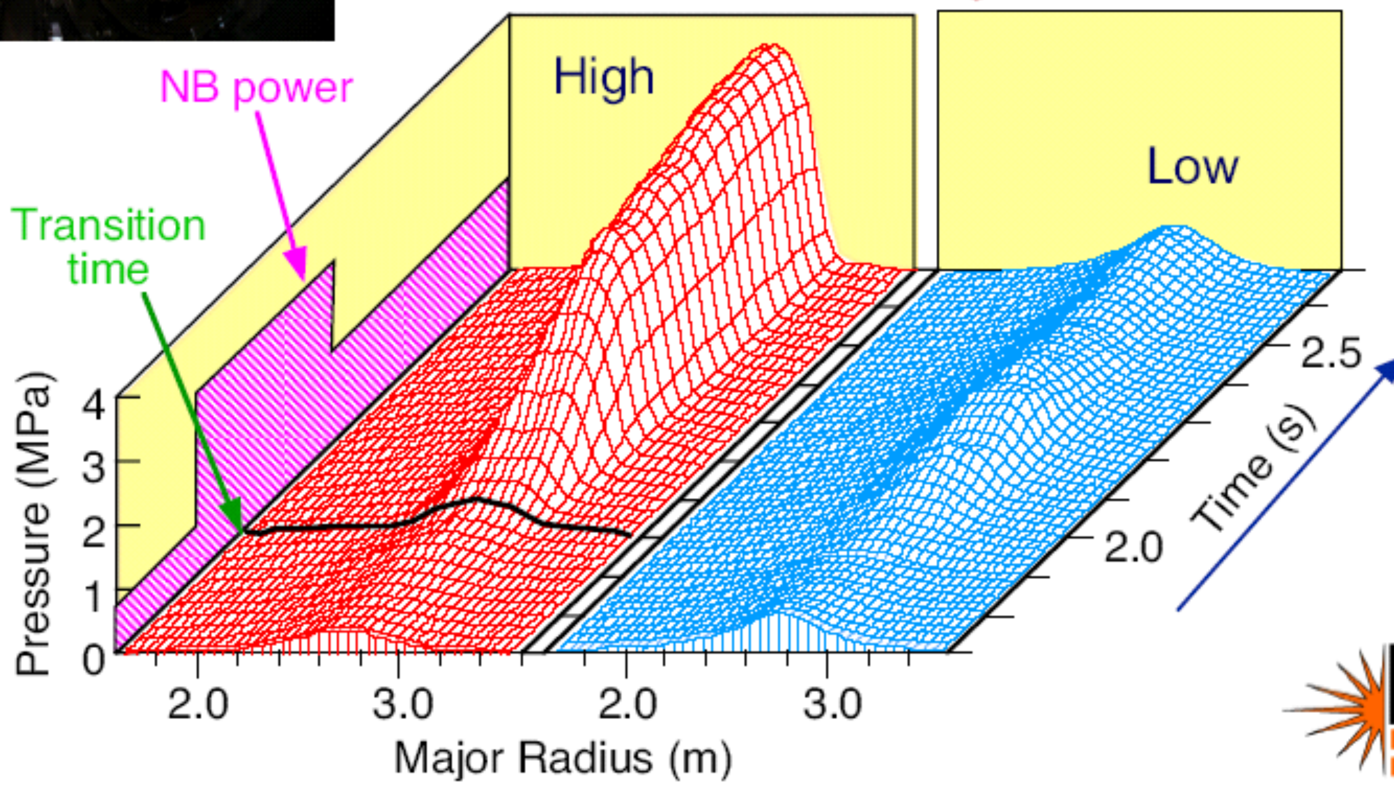
See movie “n32o6d0.8.mpg”, of a GYRO simulation.

Fascinating Diversity of Regimes in Fusion Plasmas. What Triggers Change? What Regulates Confinement?



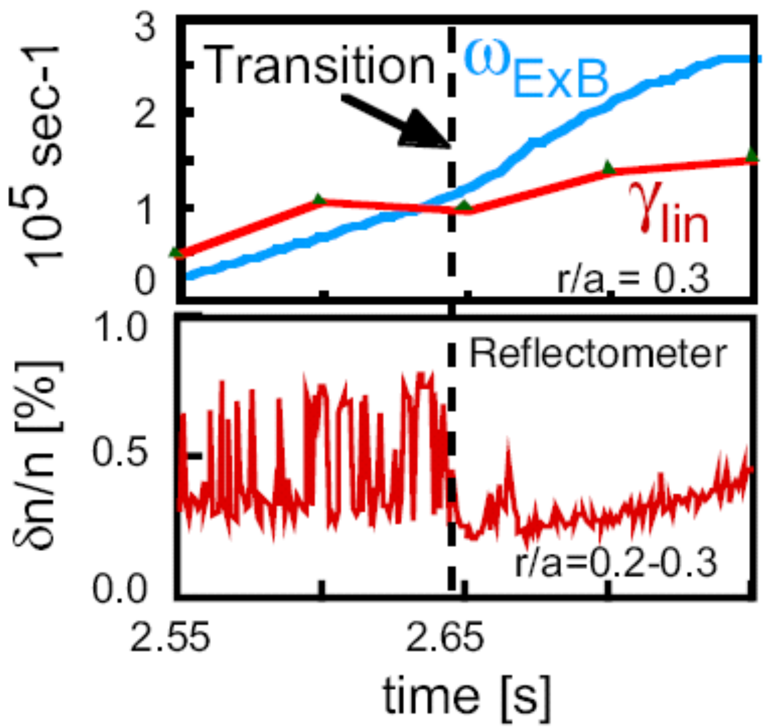
TFTR

- Two regimes with very different confinement for similar initial conditions and neutral beam heating
- Access depends on plasma heating and reducing current density on axis
- Can we attribute a difference in turbulence to these two different confinement regimes?



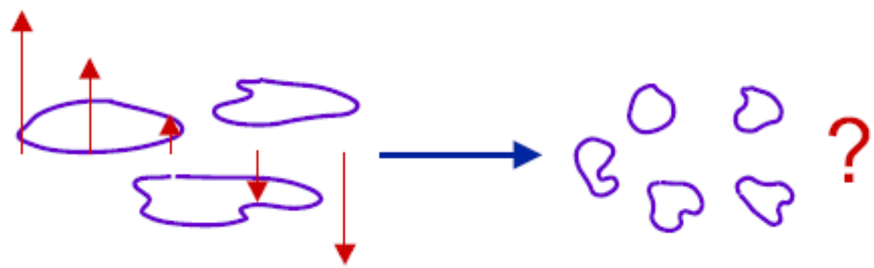
R. Nazikian et al.

Transition to Enhanced Confinement Regime is Correlated with Suppression of Core Fluctuations in TFTR



- Theory predicts fluctuation suppression when rate of shearing (ω_{ExB}) exceeds rate of growth (γ_{lin})

- Outstanding issue:
  Is suppression accompanied by radial decorrelation?



- Similar suppression observed on JET (X-mode reflectometer) and DIII-D (FIR Scattering)

Hahm, Burrell, Phys. Plas. 1995, E. Mazzucato et al., PRL 1996.

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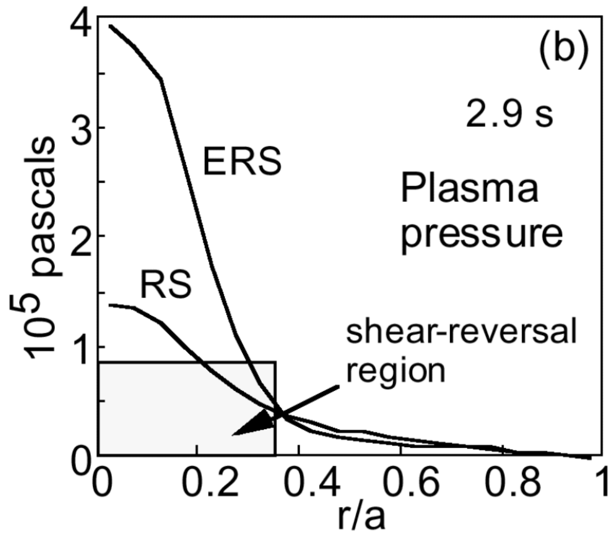
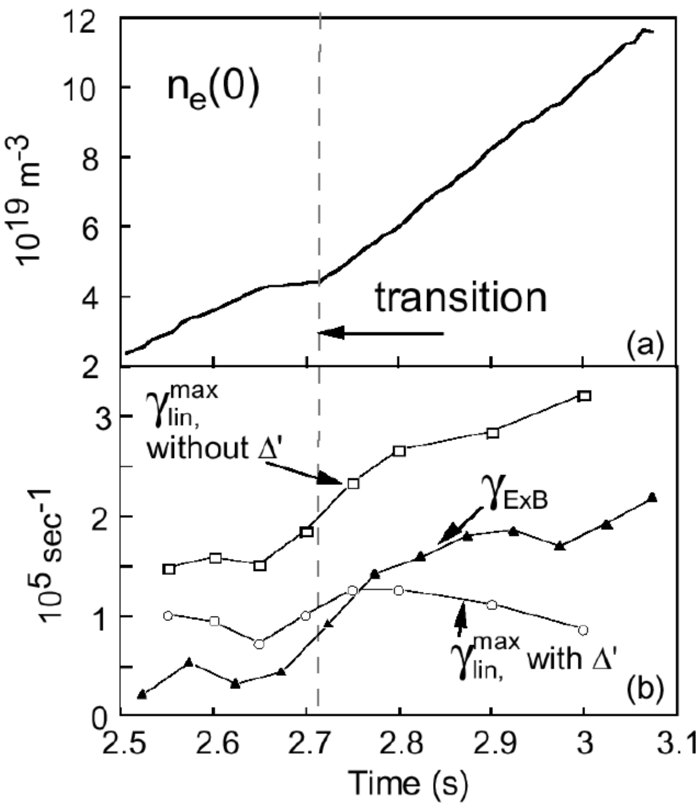
I usually denote the shearing rate as γ_{s} or γ_{ExB} instead of ω_{ExB} because it is a dissipative process and isn't like a real frequency. The shearing rate (in a simple limit of concentric circular flux surfaces) is

$$\gamma_s \approx \frac{d\mathrm{v}_{ExB,\theta}}{dr}$$

All major tokamaks show turbulence can be suppressed w/ sheared flows & negative magnetic shear / Shafranov shift





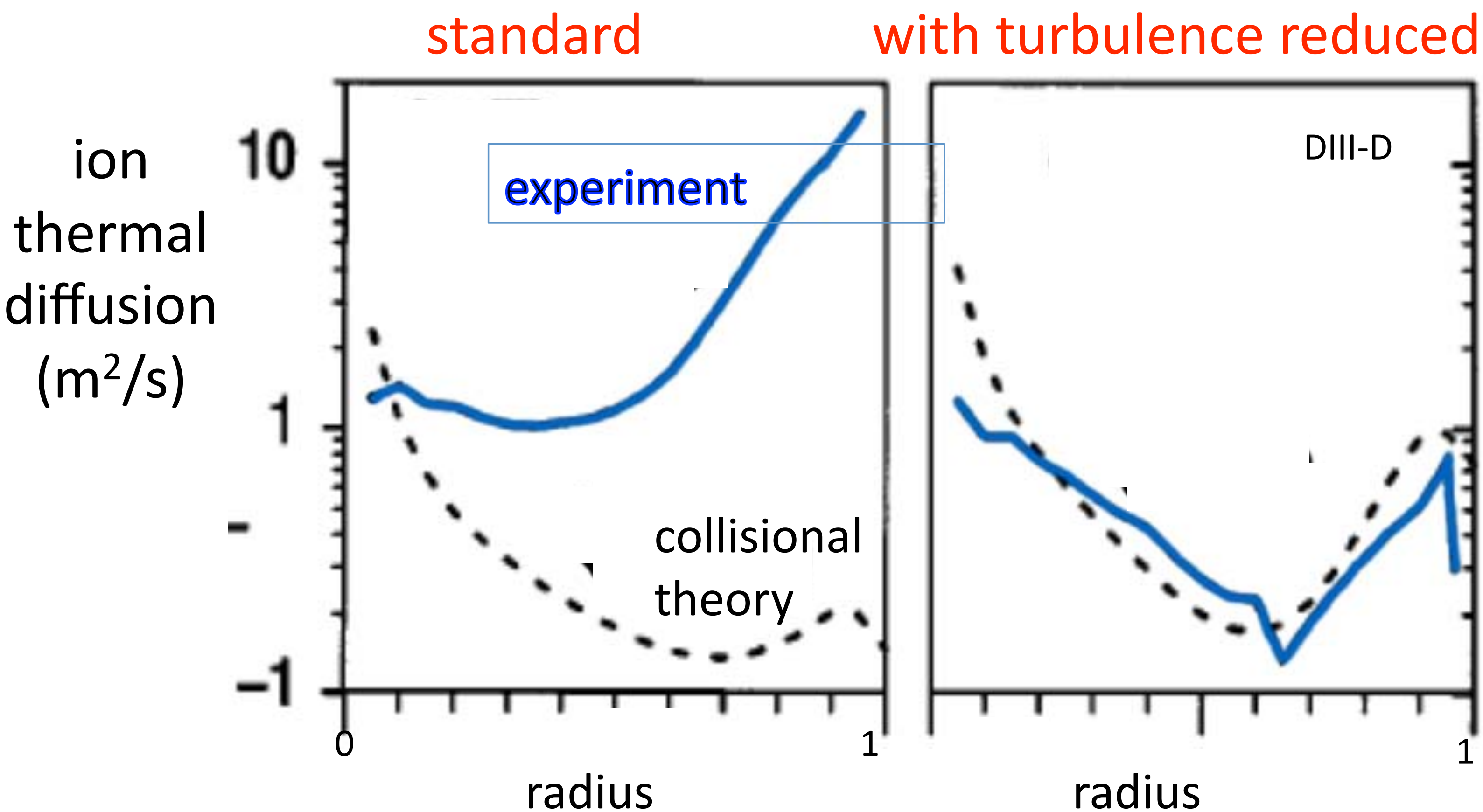
Synakowski, Batha, Beer, et.al. Phys. Plasmas 1997
& Beer, Hammett, et al., Phys. Plasmas 1997.

Internal transport barrier forms when the flow shearing rate $dv_\theta/dr >\sim$ the max linear growth rate γ_{lin}^{max} of the instabilities that usually drive the turbulence.

Shafranov shift Δ' effects (self-induced negative magnetic shear at high plasma pressure) also help reduce the linear growth rate.

Advanced Tokamak goal: Plasma pressure ~ x 2, $P_{fusion} \propto \text{pressure}^2$ ~ x 4

Turbulence suppression mechanisms really work:
Ion Transport level can be reduced to minimal collisional level in some cases.



Rough estimate of Tokamak Turbulent Diffusion

Turbulent eddies (fluctuations of electric fields that cause random ExB motions) lead to random walk diffusion. Standard assumptions are that nonlinear decorrelation must be fast enough to balance linear growth, so the correlation time $\Delta t \sim 1/\gamma \sim \sqrt{RL_p}/v_t$ and the step size $\Delta x \sim \rho_i$ (roughly, for the fastest growing modes with $k_\perp \rho_i \sim 1$). The resulting random walk diffusion coefficient:

$$D \sim \frac{(\Delta x)^2}{2\Delta t} \sim \rho_i^2 \frac{v_t}{\sqrt{RL_p}}$$

"Gyro-Bohm" scaling of diffusion

Energy confinement time $\sim$ time to diffuse to wall, $a^2 = D 2\tau_E$,

$$\tau_E = \frac{a^2}{2D} \sim \frac{a^2\sqrt{RL_p}}{\rho^2 v_t} \sim \frac{a}{v_t}\frac{a^2}{\rho^2} \sim \frac{a^3 B^2}{T^{3/2}}$$

How fast particles would be lost without magnetic field.

~10^6 in ITER

Confinement improves in larger machines and stronger magnetic field, degrades at higher temperature.

Go back to Tokamak drifts before next section.

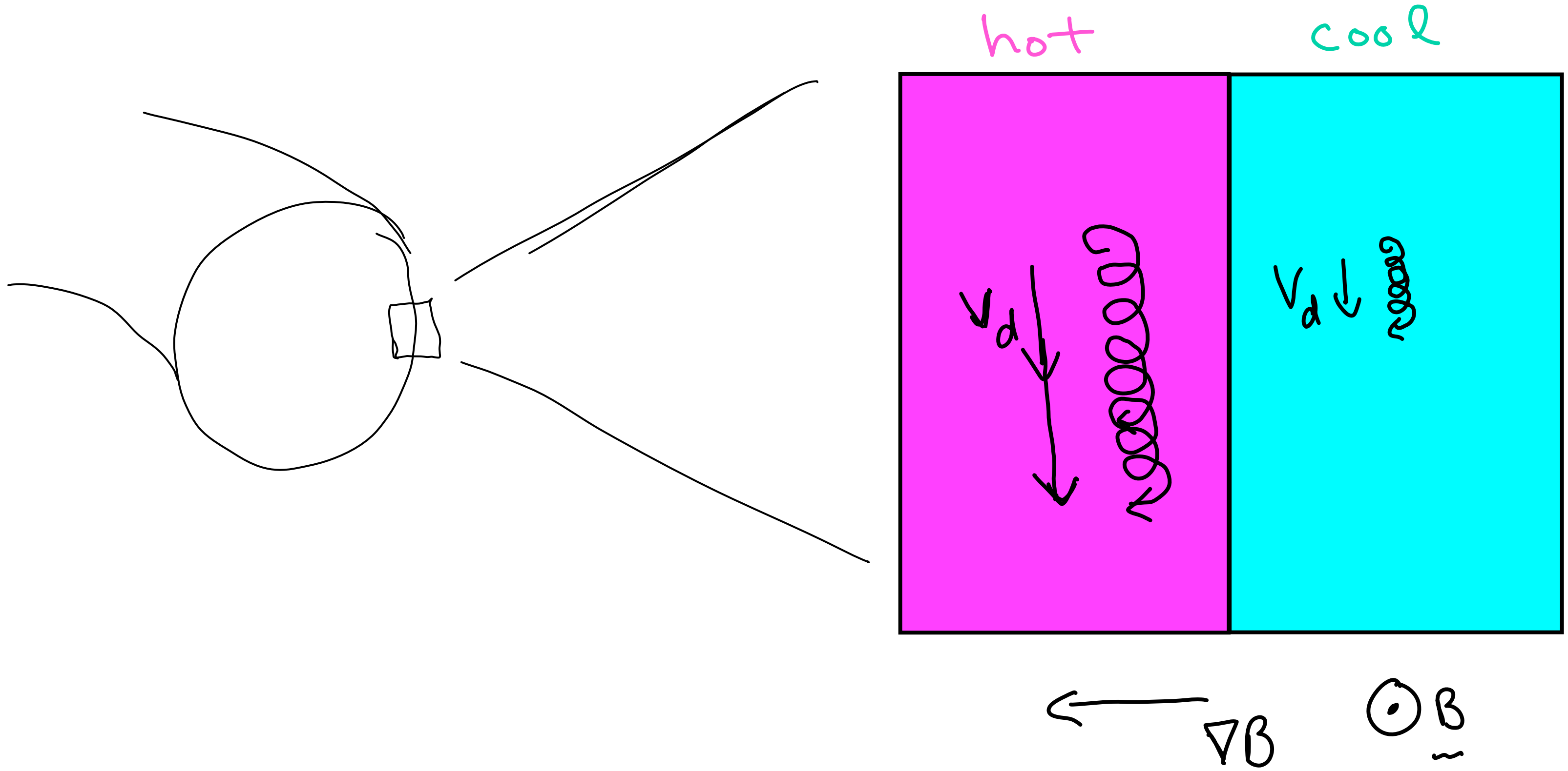
Then do single-particle picture of interchange

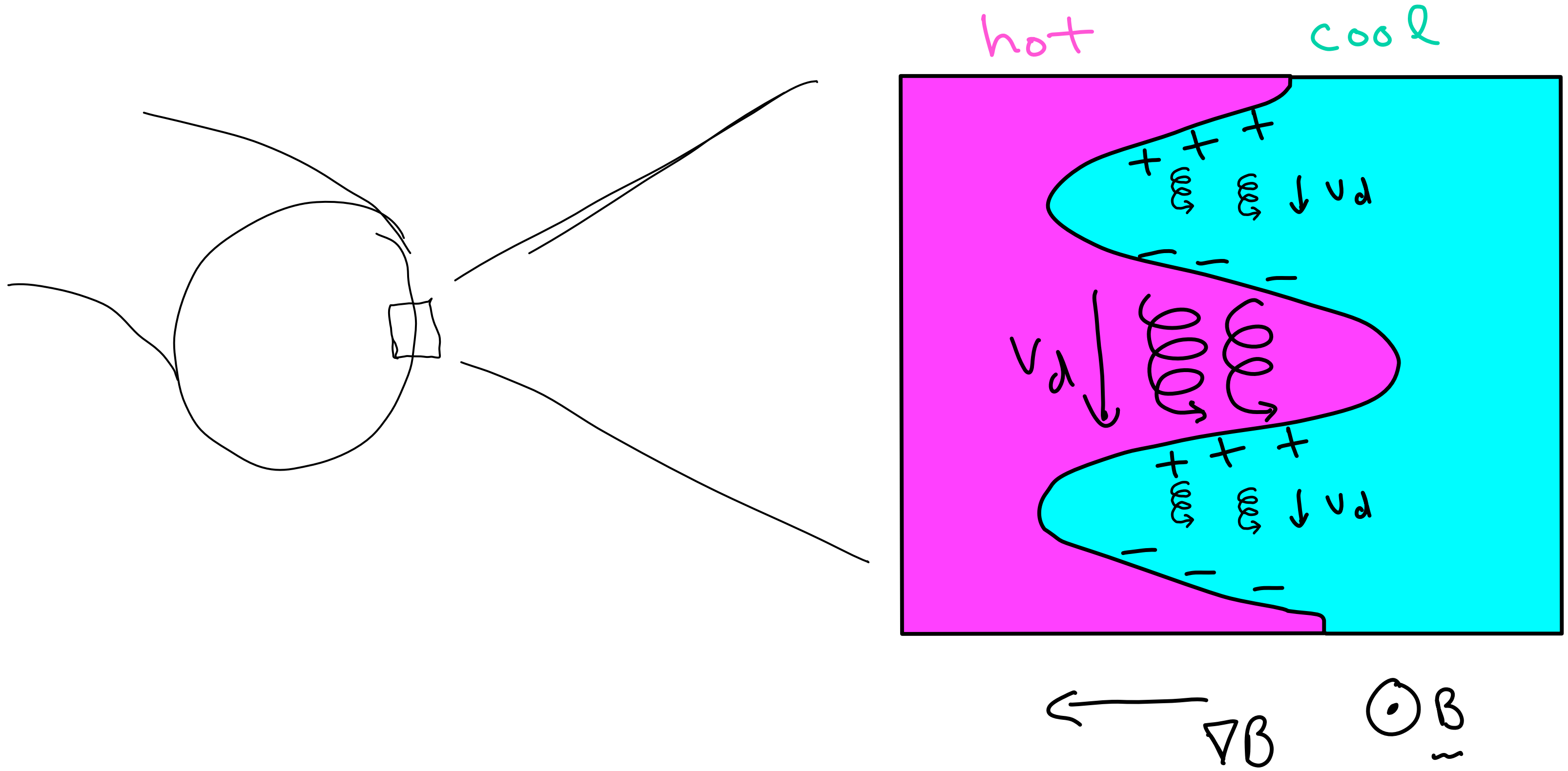
Always good to know the answer before you grind through a calculation.

Having a physical picture of an instability or other mechanism helps give insight and build confidence.

Cowley has interesting physical picture of slab η_i mode ($\eta_i = d \log(T_i) / d \log(n_e)$, which is related to the toroidal ITG mode, but it is somewhat complicated. (Corresponding dispersion relation comes from a cubic equation.)

But toroidal ITG is simpler (only involves a quadratic equation), and physical picture also simpler.


hot
cool
V_d
V_d
∇B
$\odot B$



hot
cool
v_d
v_d
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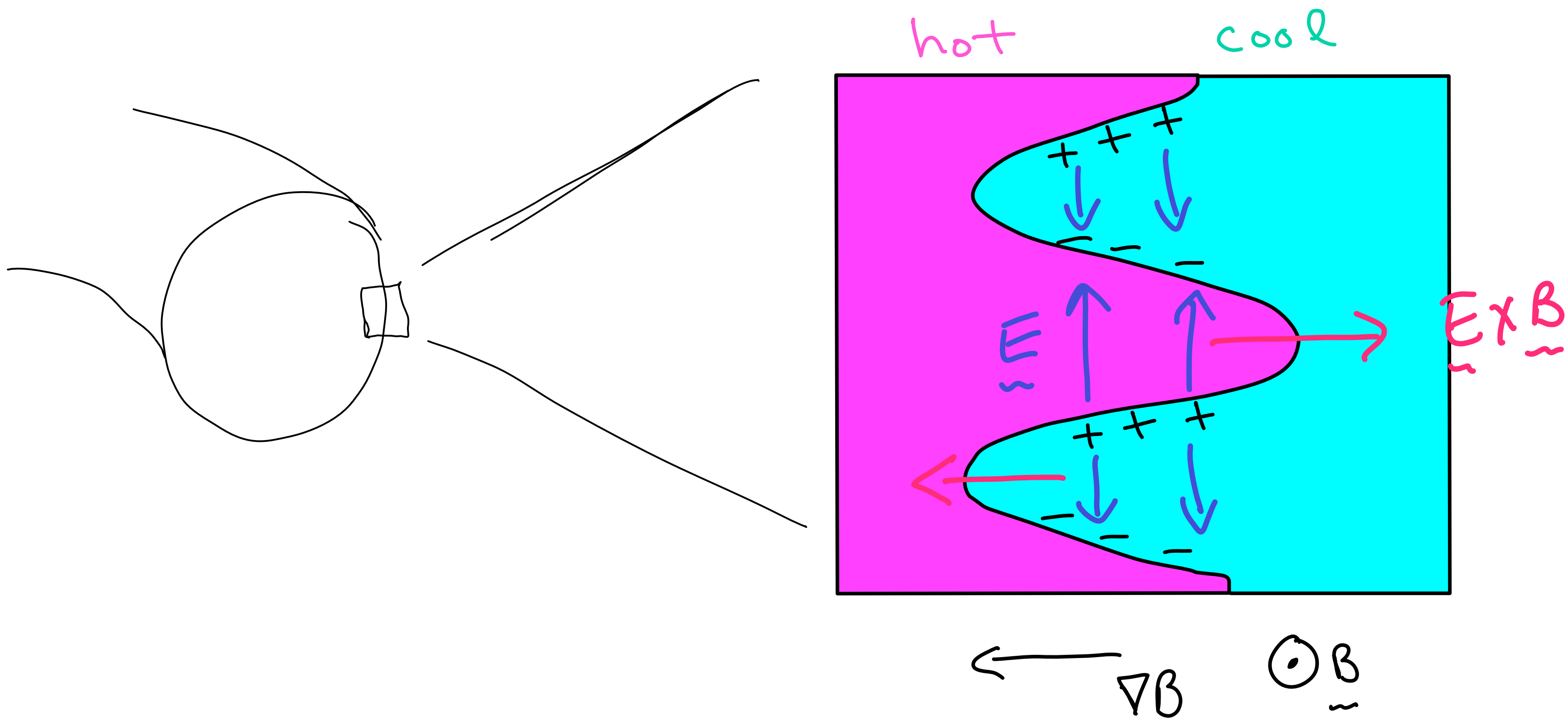
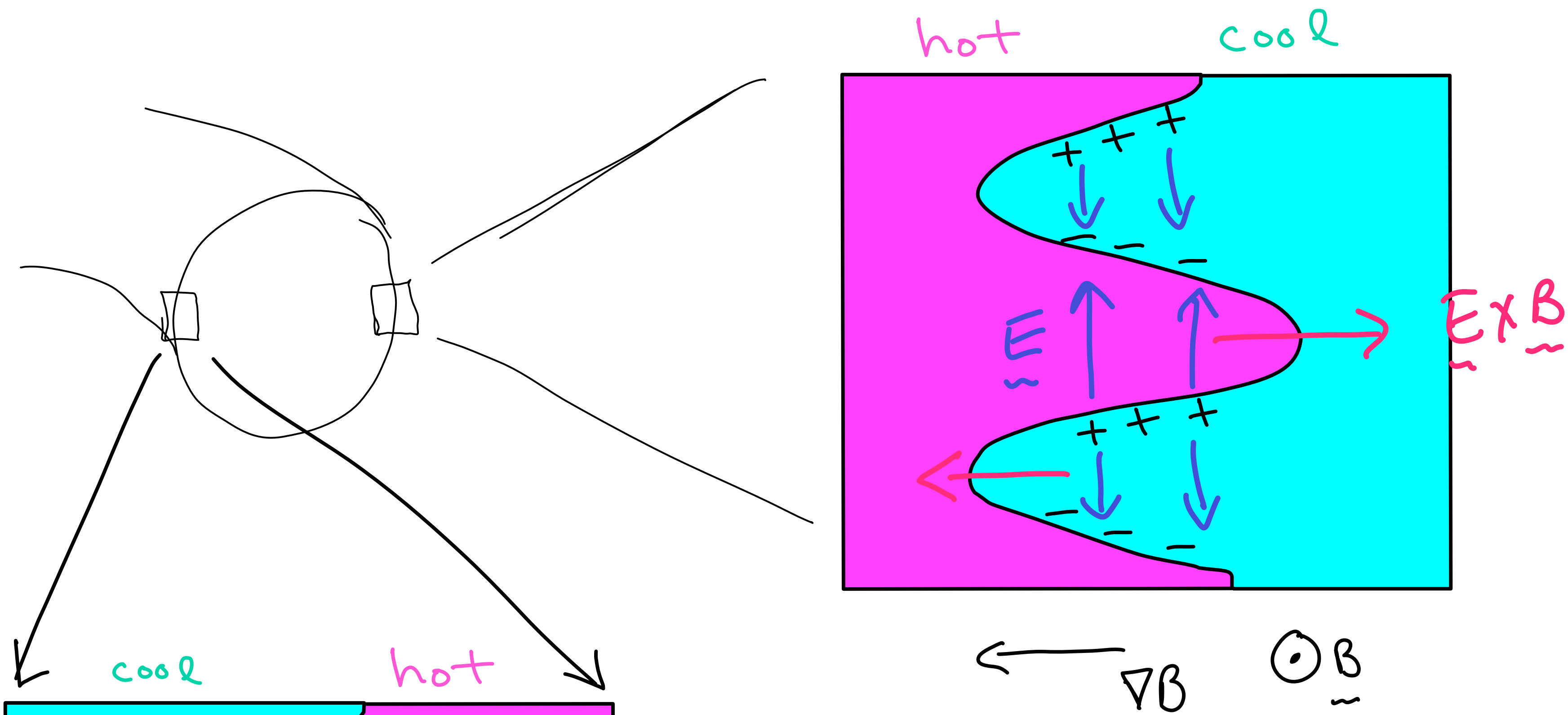


Higher energy particles $\nabla(B)$ drift faster, creates charge separation & thus $\vec{E}$ field, causes $\vec{E} \times \vec{B}$ flow that further accentuates perturbation. Positive feedback $\rightarrow$ instability.

To be more precise: the drifts cause the ion guiding center charge to become non-uniform, which would cause charge separation by itself. But an adiabatic electron response along field lines (plus the ion polarization density) will match the ion guiding center density to give quasineutrality in the end. But this still causes a + potential where the ion guiding center density perturbation is +.

Rosenbluth-Longmire 1957 picture



Can repeat this analysis on the good curvature side & find it is stable.
(Leave as exercise.)

Rosenbluth-Longmire 1957 picture

Multiscale Interaction of ETG & ITG Modes is a Challenging Research problem

- See “ETG-ki.mpg” movie (from Candy & Waltz GYRO code)
- Used sqrt(m_i/m_e) = 20 to reduce computational cost
- Left box is a 20x zoom of the lower-left corner of the big box (I think with a scaled color map).

- ITG grows more slowly, but eventually shears the ETG and reduces it to relatively low levels, in this case. However, this is for the “cyclone base case”, where the ITG chi is 50x the experimental value (because the base case was done with simplified circular geometry and ignoring background sheared flow, impurities, and beams, which reduce the ITG growth rate). Results are very problem dependent.

- See PRLs by:
- Goerler & Jenko, 2008
- Maeyama, Idomura, Watanabe et al. 2015
- Howard and Holland et al. (not PRL?), 2014-⋯

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Part 2: Rigorous derivation of ITG growth rate & threshold (in a simple limit) starting from the Gyrokinetic Eq.

Our starting point will be the electrostatic Gyrokinetic Eq. written in a Drift-Kinetic-like form for the full, gyro-averaged, guiding center density $\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\parallel}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m}E_{\parallel} - \mu \nabla_{\parallel} B + v_{\parallel}(\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E\right) \frac{\partial \bar{f}}{\partial v_{\parallel}} = 0}$$

$$\underset{\sim}{V}_E \cong - \frac{c}{B} \nabla \langle \Phi \rangle \times \hat{b} \qquad E_{\parallel} = - \hat{b} \cdot \nabla \langle \Phi \rangle$$

$$\mathbf{v}_d = \frac{v_{\parallel}^2}{\Omega}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu}{\Omega}\hat{\mathbf{b}} \times \nabla B \simeq \frac{v_{\parallel}^2 + v_{\perp}^2/2}{\Omega B^2} \underset{\sim}{B} \times \nabla B$$

$$\mu = \frac{1}{2}\frac{v_{\perp}^2}{B}$$

Gyro-averaged

$$\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t) = \langle f(\underset{\sim}{R} + \underset{\sim}{\rho}(\theta), v_{\parallel}, \mu, \theta, t) \rangle_{\theta}$$

details:

* this is not the original Drift-Kinetic Eq. of Chew, Goldberger, + Low (1956), which was for the strong E-field "MHD ordering" (see Kulsrud, Handbook of Plasma Physics, 1983)

$$V_E \sim V_t \gg V_d \sim \frac{v_\perp^2}{\Omega R} \sim v_t \frac{\rho}{R}$$

* closer to the form of the Drift-Kinetic Eq. used in neoclassical theory, where $\underset{\sim}{V_E} \sim \underset{\sim}{V_d}$ ("weak E-field")

even though $\frac{V_E}{V_t} \sim \frac{\rho}{R} \sim \varepsilon$, $\frac{V_E \cdot \nabla}{V_\parallel \hat{b} \cdot \nabla} \sim \frac{V_t \frac{\rho}{R} k_\perp}{V_t k_\parallel} \sim \frac{k_\perp \rho}{k_\parallel R}$

$$\sim 1$$

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m} E_\parallel - \mu \nabla_\parallel B + v_\parallel (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E \right) \frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

In the uniform $\underset{\sim}{B}$ slab limit, this is = to Krommes GK Eq. 4 (~ p. 11-13)

Homework: show that expanding the Boltzmann factor in Cowley's Eq. 37, & gyroaveraging to get

$$\boxed{\bar{f} = F_0 - \frac{q \langle \Phi \rangle}{T_0} F_0 + h}$$

& subst. into above GK Eq. gives exactly Cowley's (Frieman-Chen) form of the GK Eq. (Cowley Eq. 40) for $\frac{\partial h}{\partial t}$ (Use uniform B slab limit for simplicity).

[& expand in consistent assumptions: $\frac{q\langle \Phi \rangle}{T} \ll 1$ but

$$F_0 \nabla_\perp \frac{q \langle \Phi \rangle}{T_0} \sim \nabla_\perp F_0 \qquad F_0 \nabla_\perp \frac{q \langle \Phi \rangle}{T_0} \sim \nabla_\perp F_0$$]

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m} E_\parallel - \mu \nabla_\parallel B + v_\parallel (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E \right) \frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

Homework: show that substituting the gyro-average of Cowley's Eq. 37:

$$\bar{f} = F_0 - q \frac{\langle \Phi \rangle}{T_0} F_0 + h$$

(Straight uniform $\underset{\sim}{B}$ limit for simplicity)

$$\frac{\partial h}{\partial t} - \frac{q}{T_0} \frac{\partial \langle \Phi \rangle}{\partial t} F_0 + v_\parallel \hat{b} \cdot \nabla h + \underset{\sim}{v}_E \cdot \nabla h + \underset{\sim}{v}_E \cdot \nabla \left(F_0 \left(1 - \underbrace{q \frac{\langle \Phi \rangle}{T_0}}_{\text{drop}} \right) \right)$$

$$- v_\parallel \hat{b} \cdot \nabla \langle \Phi \rangle \frac{q}{T_0} F_0 \qquad \underbrace{- \frac{q}{m} \hat{b} \cdot \nabla \langle \Phi \rangle \frac{\partial h}{\partial v_\parallel}}_{\text{drop}}$$

$$- \frac{q}{m} \hat{b} \cdot \nabla \langle \Phi \rangle \frac{\partial F_0}{\partial v_\parallel} \left(1 - \underbrace{q \frac{\langle \Phi \rangle}{T_0}}_{\text{drop}} \right) = 0$$

These 2 terms cancel

Use $\frac{\partial F_0}{\partial v_\parallel} = - \frac{m v_\parallel}{T_0} F_0$

Homework Solution outline

$$\boxed{\frac{\partial \overline{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla \overline{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \overline{f}}{\partial v_{\|}} = 0}$$

Linearize: $\overline{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

$\frac{\partial}{\partial t} = 0 \quad \underset{\sim}{E} = 0$

$$(v_{\|}\hat{b} + \underset{\sim}{v}_d)\cdot\nabla F_0 - \mu\nabla_{\|}B\frac{\partial F_0}{\partial v_{\|}} = 0$$

Basically says $F_0 = \text{const.}$ along trajectories of banana orbits or passing orbits in a tokamak.

General Equilibrium solution could be an arbitrary function of the constants of the motion (E, μ, P_ϕ) where

$$E = \tfrac{1}{2}mv_{\|}^2 + m\mu B$$

& P_ϕ = canonical angular momentum

But if we neglect $\frac{|\underset{\sim}{v}_d|}{v_{\|}} \sim \frac{\rho}{R}$ get simpler Eq:

$$v_{\parallel} \hat{b} \cdot \nabla F_0 - \mu \left(\hat{b} \cdot \nabla B \right) \frac{\partial F_0}{\partial v_{\parallel}} = 0$$

Will consider Equilibrium of the form:

$$F_0(\underset{\sim}{R}, v_{\parallel}, \mu) \propto \frac{n_0(\psi)}{T_0^{3/2}(\psi)} e^{-\frac{m\left(\frac{1}{2} v_{\parallel}^2 + \mu B(\underset{\sim}{x})\right)}{T(\psi)}} \propto e^{-\frac{E}{T}}$$

Exercise: Plug this in to the previous Eq. & show it is a solution.

(This is a Maxwellian equilibrium that depends only on energy. Can generalize to bi-Maxwellian equilibria that depend on energy and mu, the constants of the motion, or more generally, any arbitrary function of $E = (1/2)\, m\, v_{||}^2/2 + m\, \mu\, B(z)$ and of μ. Collisions often ignorable for fast fluctuations tilde f, but important for equilibrium F_0.)

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_\parallel - \mu\nabla_\parallel B + v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

Linearize: $\bar{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

Next order Eq:

$$\frac{\partial \tilde{f}}{\partial t} + \left(v_\parallel \hat{b} + \underset{\sim}{v}_d\right)\cdot\nabla\tilde{f} - \mu\nabla_\parallel B\,\frac{\partial \tilde{f}}{\partial v_\parallel} = -\underset{\sim}{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_\parallel + v_\parallel\left(\hat{b}\cdot\nabla\hat{b}\right)\cdot\underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_\parallel}$$

$$\left(-i\omega + i v_\parallel k_\parallel + i\,\underset{\sim}{v}_d\cdot\underset{\sim}{k}_\perp\right)\tilde{f} = -\underset{\sim}{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_\parallel + v_\parallel\left(\hat{b}\cdot\nabla\hat{b}\right)\cdot\underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_\parallel}$$

(focus on bad-curvature region near $\theta=0$ where $\nabla_\parallel B=0$,
Fourier transform small-scale fluctuation taking coefficients locally fixed)

Important Subtlety: $F(\underset{\sim}{R}, v_\parallel, \mu, t)$ so

$$-\underset{\sim}{v}_E\cdot\nabla F_0 = -\underset{\sim}{v}_E\cdot\nabla\Big|_{v_\parallel,\mu,t} F_0$$

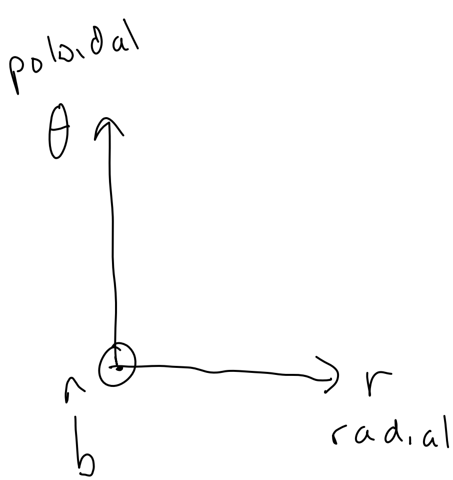
using $$F_0 \propto \frac{n_0(r)}{T_0^{3/2}(r)}\, e^{-\frac{\left(\frac{1}{2}mv_\parallel + m\mu B(\underset{\sim}{x})\right)}{T_0(r)}}$$

will give terms proportional to ∇n_0, ∇T_0, & $\mu\nabla B$

∇n_0 terms: $$-\underset{\sim}{v}_E\cdot\nabla F_0 \Rightarrow +\frac{c}{B}\left(\nabla\Phi\times\hat{b}\cdot\frac{\nabla n_0}{n_0}\right)F_0$$

$$\frac{\nabla n_0}{n_0} = -\frac{\hat{r}}{L_n}$$

$$= -\frac{c}{B}\nabla\Phi\times\hat{b}\cdot\hat{r}\,\frac{1}{L_n}F_0$$

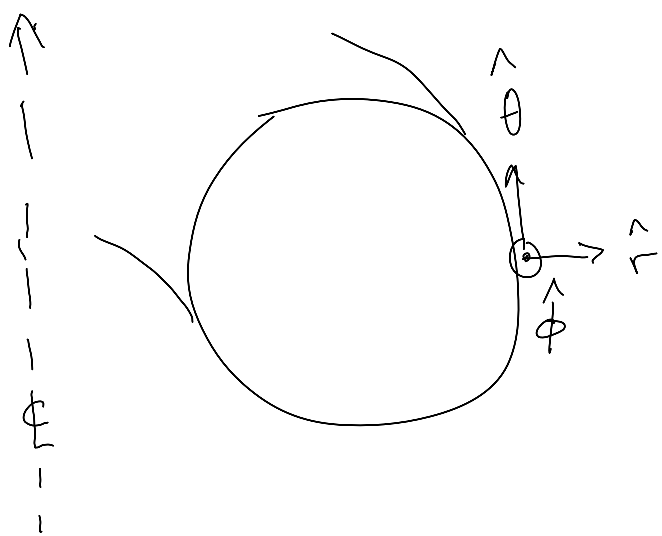


$$= -\frac{c}{B}\,ik_\theta\,\Phi\,\frac{1}{L_n}F_0$$

$$= +i\omega_*\frac{e\Phi}{T_0}F_0$$

$$\omega_{*i} \equiv -\frac{cT_i}{e_iB}\frac{k_\theta}{L_n} = -\omega_{*e}\frac{T_i}{T_e}$$
$$\equiv -k_\theta\rho_i\frac{v_{ti}}{L_n}$$

Note on sign conventions:



With $\underset{\sim}{B}$ field out of the page, the ∇B drift for ions is downward

$$\underset{\sim}{V}_d \approx -\hat{\theta}\, v_t \frac{\rho}{R} \quad (\text{at } \theta = 0)$$

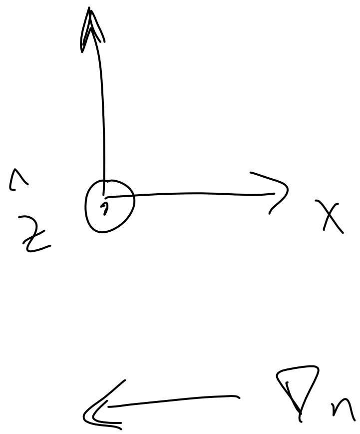
defining $\omega_{dv} = \underset{\sim}{k} \cdot \underset{\sim}{V}_d$

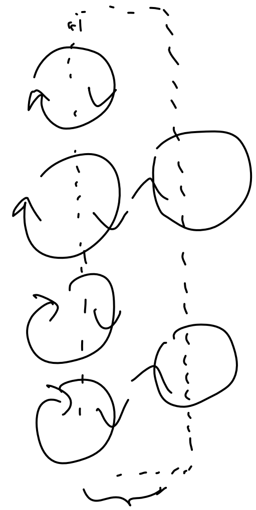
gives convention used in Beer's thesis!

$$\omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_d = -k_\theta \rho v_t / R$$

More on Sign Conventions





With $\underset{\sim}{B}$ out of page, the diamagnetic flow V_{*i} is downward if ∇n is inward. Thus

$$\omega_{*i} \equiv \underset{\sim}{k}\cdot V_{*i} = -k_\theta v_t \frac{\rho}{L_n}$$

$$= -\frac{cT}{eB}\frac{k_\theta}{L_n}$$

(Spitzer's resolution of Spitzer's paradox: fluid flows ≠ particle guiding center drifts.)

(All quantities here, including charge e, evaluated for the species for this f.)

(Back to RHS of linearized GK Eq., 4 slides back)

$$RHS = \underbrace{-\underset{\sim}{v}_E \cdot \nabla F_0}_{\text{part of this}} - \Big(\underbrace{\frac{q}{m} E_{\parallel} + v_{\parallel}(\hat{b}\cdot\nabla\hat{b})\cdot \underset{\sim}{v}_E\Big)\frac{\partial F_0}{\partial v_{\parallel}}}$$

part of this

$$\propto -\frac{c}{B}\nabla\Phi \times \hat{b} \cdot \mu \nabla B \qquad\qquad \propto + v_{\parallel}^2 (\hat{b}\cdot\nabla\hat{b}) \cdot \left(\frac{\hat{b}\times\nabla\Phi}{B}\right)$$

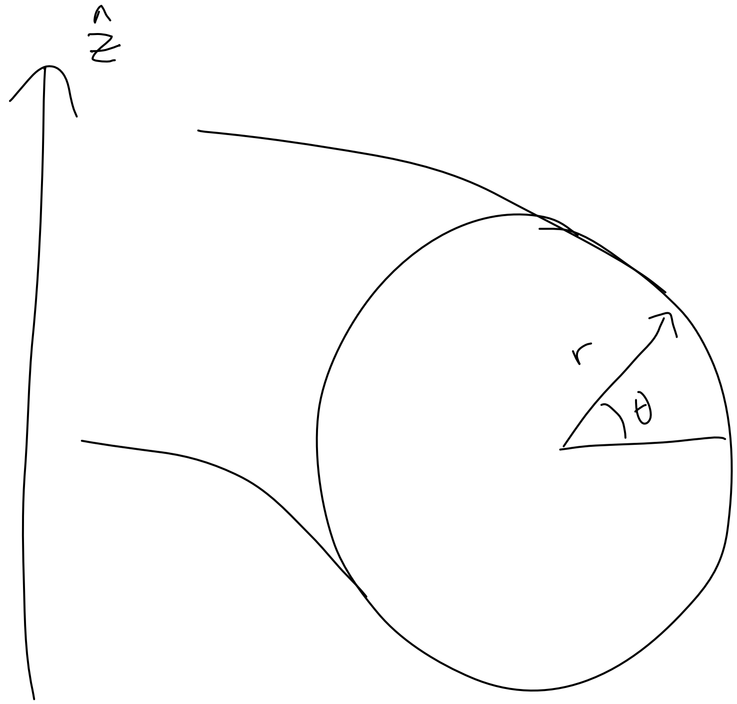
$$\propto -\nabla\Phi \cdot \Big[\mu \, \hat{b} \times \nabla B + v_{\parallel}^2 \, \hat{b} \times (\hat{b}\cdot\nabla\hat{b})\Big]$$

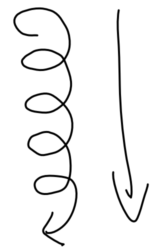
∇B + curvature drift

$$RHS = +i\left(\omega_{*v}^T - \omega_{dv} - k_{\parallel} v_{\parallel}\right)\frac{e\Phi}{T_0} F_0$$

$$\omega_{*v}^T = \omega_*[1 + \eta(v_{\parallel}^2/2v_t^2 + \mu B/v_t^2 - 3/2)] \qquad\qquad \omega_{dv} = \omega_d(v_{\parallel}^2 + \mu B)/v_t^2$$

$$\omega_* = -k_\theta \rho \frac{v_t}{L_n} \qquad \eta = \frac{L_n}{L_T} \qquad \omega_d = -\frac{v_t}{R}\rho\left(k_\theta \cos\theta + k_r \sin\theta\right)$$





downward

$\underset{\sim}{v}_d$ from ∇B + curvature drift

$$\omega_d = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$$

$$= -\frac{v_t \rho}{R}\left(k_\theta \cos\theta + k_r \sin\theta\right)$$

will focus on $\theta \simeq 0$ here

(where bad-curvature drive is the strongest)

$$\left(-i\omega + i v_\parallel k_\parallel + i \underset{\sim}{v_d} \cdot \underset{\sim}{k_\perp}\right) \tilde{f} = -\underset{\sim}{v_E} \cdot \nabla F_0$$

$$- \left(\frac{q}{m} E_\parallel + v_\parallel \left(\hat{b} \cdot \nabla \hat{b}\right) \cdot \underset{\sim}{v_E}\right) \frac{\partial F_0}{\partial v_\parallel}$$

subst. for RHS

$$\left(-i\omega + i v_\parallel k_\parallel + i\omega_{dv}\right) \tilde{f} = -i\left(-\omega_{*v}^T + \omega_{dv} + k_\parallel v_\parallel\right) \frac{e\Phi}{T_0} F_0$$

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^T + \left(k_\parallel v_\parallel + \omega_{dv}\right)}{\omega - \left(k_\parallel v_\parallel + \omega_{dv}\right)} \frac{e\Phi}{T_0} F_0}$$

Note: recover Boltzmann response when $k_\parallel v_\parallel$ &/or ω_{dv} large

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^T + (k_\parallel v_\parallel + \omega_{dv})}{\omega - (k_\parallel v_\parallel + \omega_{dv})} \frac{e\Phi}{T_0} F_0}$$

Look for modes with

$$k_\parallel v_{ti} \ll \omega, \omega_{*v}^T, \omega_{dv} \ll k_\parallel v_{te}$$

(slab "η_i" version of ITG requires finite $k_\parallel v_{ti}$, but not toroidal version).

assume Boltzmann electrons

Quasineutrality: $\tilde{n}_e = \tilde{n}_i$

(additional polarization contribution to density gives $k_\perp^2 \rho_i^2$ corrections but not critical for basic ITG.)

$$\boxed{n_{e0} \frac{e\Phi}{T_e} = \int d^3v \frac{-\omega_{*v}^T + \omega_{dv}}{\omega - \omega_{dv}} F_0 \frac{e\Phi}{T_{i0}}}$$

$$n_0 \frac{e\Phi}{T_{e0}} = n_0 \frac{e\Phi}{T_{i0}} \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega - \omega_{dv}}$$

"Cold plasma" or "fast wave" approx. $\omega \gg \omega_{dv}$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_* [1 + \eta (v_\parallel^2 / 2v_t^2 + \mu B / v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} = \int d^3v \, \frac{F_0}{n_0} \, \omega_d \left(v_\parallel^2 + \frac{1}{2} \underbrace{v_\perp^2}_{= v_x^2 + v_y^2} \right) / v_t^2$$

$$= 2 \omega_d$$

Using useful I.D. for Maxwellian F_0:

$$\langle v_x^{2n} \rangle = \int d^3v \, \frac{F_0}{n_0} \, v_x^{2n} = v_t^{2n} \underbrace{(2n-1)!!}_{(2n-1)(2n-3)(2n-5)\cdots 5 \cdot 3 \cdot 1}$$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \underbrace{\mu B}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t/R \qquad \omega_* = -k_\theta \rho v_t / L_n \qquad = \frac{1}{2} v_\perp^2 = \frac{1}{2}\left(v_x^2 + v_y^2\right)$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_*^T = \omega_* \left(1 + \eta \underbrace{\left(\frac{1}{2} + 1 - \frac{3}{2} \right)}_{0} \right)$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv}^2 = \int d^3v \, \frac{F_0}{n_0} \omega_d^2 \left[v_\parallel^4 + 2 v_\parallel^2 \, \frac{1}{2} v_\perp^2 + \frac{1}{4} \left(v_x^2 + v_y^2 \right)^2 \right] \frac{1}{v_t^4}$$

$$= \omega_d^2 \left[3 + 2 \cdot \frac{1}{2}(1+1) + \frac{1}{4} \left(\frac{\langle v_x^4 + 2 v_x^2 v_y^2 + v_y^4 \rangle}{v_t^4} \right) \right]$$

$$= \omega_d^2 \left[5 + \frac{1}{4}(8) \right] = 7\, \omega_d^2$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \underbrace{\mu B}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t/R \qquad \omega_* = -k_\theta \rho v_t/L_n \qquad = \frac{1}{2} v_\perp^2 = \frac{1}{2}\left(v_x^2 + v_y^2\right)$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T = \omega_d \omega_* \Big\{ 2$$

$$+ \eta \int d^3v \, \frac{F_0}{n_0} \, \frac{\left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right)}{v_t^2} \, \frac{\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_\perp^2 - \frac{3}{2} v_t^2 \right)}{v_t^2} \Big\}$$

$$= \omega_d \omega_* \Big\{ 2 + \eta \Big[\frac{1}{2} 3 + \frac{1}{2} 2 - \frac{3}{2} + \frac{1}{2} \cdot 2 \cdot \frac{1}{2} + \frac{1}{4} 8$$

$$- \frac{1}{2} \cdot 2 \cdot \frac{3}{2} \Big] \Big\}$$

$$\int d^3v\, \frac{F_0}{n_0}\, \omega_{dv}\, \omega_*^T = \omega_d \omega_* \left\{ 2 + \eta \left[\cancel{\tfrac{1}{2} 3} + \cancel{\tfrac{1}{2} 2} - \cancel{\tfrac{3}{2}} + \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}} + \tfrac{1}{4} 8 - \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{3}{2}} \right] \right\}$$

$$= \omega_d \omega_* 2 (1 + \eta)$$

Combine results from last 2 pages:

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2 \frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7 \frac{\omega_d^2}{\omega^2} - 2 \frac{\omega_d \omega_*}{\omega^2} (1 + \eta)}$$

This defines a dispersion relation ω vs. $\underset{\sim}{k}$

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2\frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7\frac{\omega_d^2}{\omega^2} - 2\frac{\omega_d\omega_*}{\omega^2}(1+\eta)}$$

Consider the flat density limit: $\nabla n \to 0$, but $\nabla T \neq 0$

$$\omega_* = -k_\theta \rho \frac{v_t}{L_n} \to 0 \qquad \eta = \frac{\frac{1}{T}\nabla T}{\frac{1}{n}\nabla n} = \frac{L_n}{L_T} \to \infty$$

$$\omega_*\eta = -k_\theta \rho \frac{v_t}{L_n}\frac{L_n}{L_T} \equiv \bar{\omega}_{*T}$$

$$\omega^2 \frac{T_{i0}}{T_{e0}} - 2\omega_d\omega + 2\omega_d\bar{\omega}_{*T} - 7\omega_d^2 = 0$$

$$\boxed{\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d\bar{\omega}_{*T} - 7\omega_d^2\right)}}{2\left(T_{i0}/T_{e0}\right)}}$$

From last page:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \overline{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}$$

Consider large temperature gradient limit: $\omega_{*T} \propto \nabla T$

Growth rate:

$$\gamma = \frac{\sqrt{2\omega_d \overline{\omega}_{*T}}}{\sqrt{T_{i0}/T_{e0}}} = \frac{\sqrt{2}\, k_\theta \rho_i}{\sqrt{T_{i0}/T_{e0}}} \frac{v_{ti}}{\sqrt{R L_T}}$$

Fundamental scaling of bad-curvature driven instabilities.

Go back to general D.R.:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \bar{\omega}_{*T} - 7\omega_d^2\right)}}{2\left(T_{i0}/T_{e0}\right)}$$

$$= \frac{2\omega_d \pm \sqrt{\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)\omega_d^2 - 8\frac{T_{i0}}{T_{e0}}\omega_d \bar{\omega}_{*T}}}{2\left(T_{i0}/T_{e0}\right)}$$

Instability exists if

$$8\frac{T_{i0}}{T_{e0}}\omega_d \bar{\omega}_{*T} > \omega_d^2\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)$$

$$\frac{1}{R}\frac{1}{L_T} > \frac{1}{R^2}\left(\frac{1}{2}\frac{T_{e0}}{T_{i0}} + \frac{1}{2}7\right)$$

$$\star \quad \boxed{\frac{R}{L_T} > \frac{1}{2}\left(7 + \frac{T_{e0}}{T_{i0}}\right)}$$

Compare w/ Romanelli 1990 (Eq. 12):

$$\eta_i = (\tfrac{5}{3} + \tau/4)2\epsilon_n$$

or $$\frac{L_n}{L_T} = \left(\frac{5}{3} + \frac{1}{4}\frac{T_e}{T_i}\right) 2 \frac{L_n}{R}$$

$$\boxed{\frac{R}{L_{Tcrit}} = \frac{10}{3} + \frac{1}{2}\frac{T_{eo}}{T_{io}}}$$

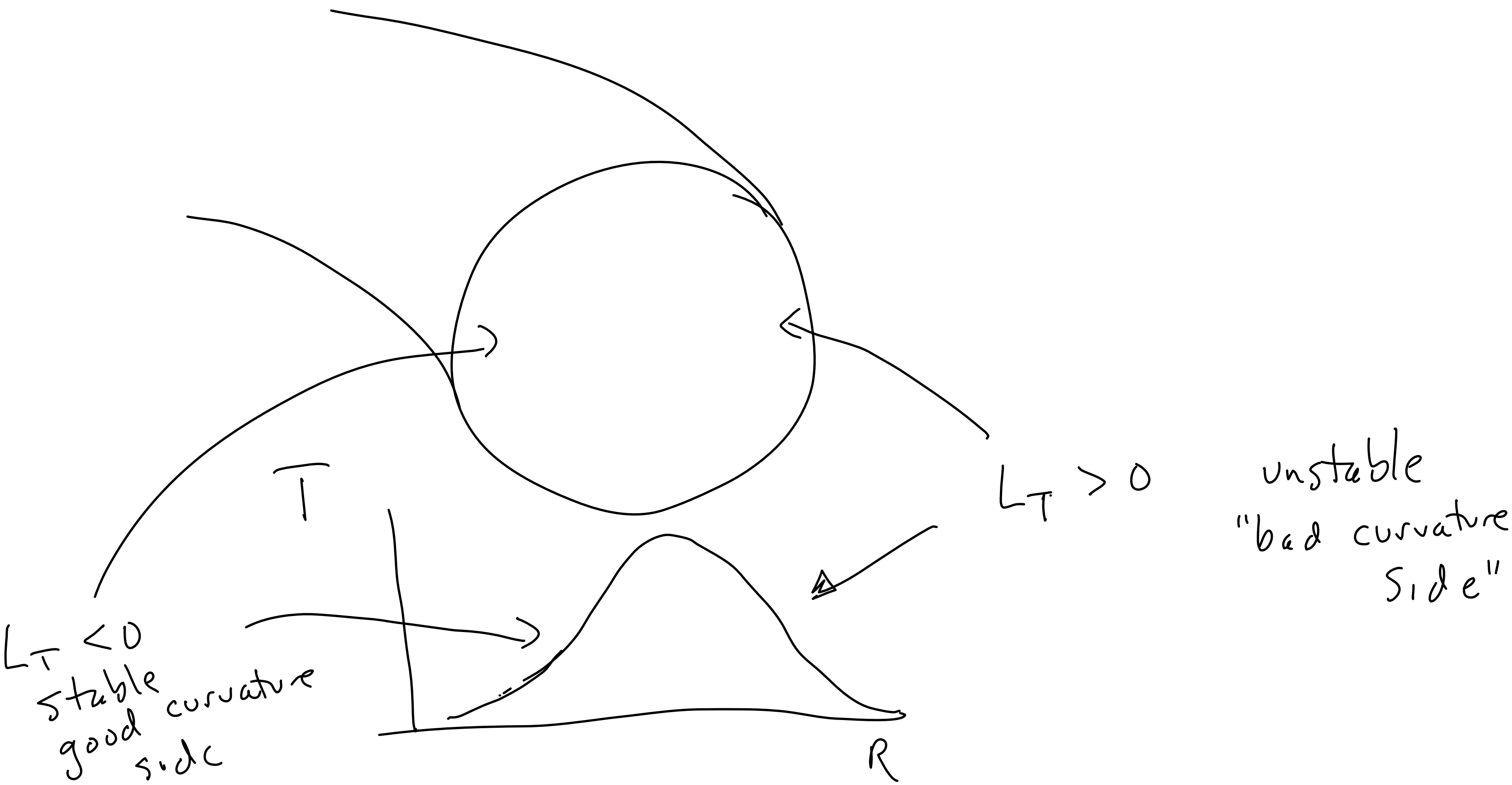
$$= 3.33 + 0.5 \frac{T_{eo}}{T_{io}}$$

vs. my $$\frac{R}{L_{Tcrit}} = 3.5 + 0.5\frac{T_{eo}}{T_{io}}$$

Very close. Diff. is presumably because Romanelli simplifies ω_{Dv} (See after his Eq. 6)

Note there is an instability only if $\omega_d \overline{\omega}_{*T} > 0$

$$\omega_d \overline{\omega}_{*T} = (k_\theta \rho)^2 \frac{v_t^2}{R L_T} \qquad \frac{1}{L_T} \equiv -\frac{1}{T}\frac{\partial T}{\partial R}$$



Why does this get the $\frac{T_{i0}}{T_{e0}}$ dependence of

$\frac{R}{L_{T\,crit}}$ wrong?

More accurate: $\boxed{\frac{R}{L_T} > \frac{R}{L_{T\,crit}} = \frac{4}{3}\left(1 + \frac{T_{i0}}{T_{e0}}\right)}$

Because near marginal stability, the expansion of the resonant denominator

$$\frac{1}{\omega - \omega_{dv}} \approx \frac{1}{\omega}\left(1 + \frac{\omega_{dv}}{\omega} + \dots\right)$$

breaks down, since $\omega \sim \omega_d$ near marginal stability...

More general result for threshold for instability:

$$\frac{R_0}{L_{T\,crit}} = \mathrm{Max}\left[\left(1 + \frac{T_i}{T_e}\right)\left(1.33 + 1.91\,\frac{\hat{s}}{q}\right)\left(1 - 1.5\,\frac{r}{R_0}\right)\left(1 + 0.3\,\frac{r\,d\kappa}{dr}\right),\ 0.8\,\frac{R_0}{L_n} \right]$$

Found by fits to lots of GS2 Gyrokinetic stability calculations (Jenko, Dorland Hammett, PoP 2001), guided by previous analytic results (Romanelli, Hahm + Tang) in some limits.

ITG References

- Online links to some of these papers are at http://w3.pppl.gov/~hammett/papers/

- Mike Beer's Ph.D. Thesis 1995
  http://w3.pppl.gov/~hammett/collaborators/mbeer/afs/thesis.html
  Ph.D. dissertations often provide good tutorials on a topic. Presents tutorial on fundamentals and physical pictures of ITG mode, and the first comprehensive 3D gyrofluid simulations of ITG and TEM turbulence in realistic toroidal geometry. (Gyrofluid equations include models of FLR & kinetic effects like Landau damping.) Documents the important role of turbulence-generated zonal flows in saturating toroidal ITG turbulence, and the major reduction of ITG turbulence by using a proper adiabatic electron response that does not respond to zonal electric fields with $E_{||}=0$ (also shown in slab limit in Dorland's earlier thesis).

- My derivation of the linear dispersion relation here follows the similar calculation in Beer's thesis above, and Beer and Hammett, "Toroidal gyrofluid equations for simulations of tokamak turbulence", Phys. Plasmas 3, 4046 (2011) https://w3.pppl.gov/~hammett/gyrofluid/papers/1996/beer-tor96.pdf

- Early history:
  - slab eta_i mode: Rudakov and Sagdeev, 1961
  - Sheared-slab eta_i mode: Coppi, Rosenbluth, and Sagdeev, Phys. Fluids 1967
  - Toroidal ITG mode: Coppi and Pegoraro 1977, Horton, Choi, Tang 1981, Terry et al. 1982, Guzdar et al. 1983… (See Beer's thesis)
- Romanelli & Briguglio, Phys. Fluids B 1990
- Biglari, Diamond, Rosenbluth, Phys. Fluids B 1989
  These two are detailed analytic papers on ITG dispersion relations and mixing-length estimates of turbulent transport. The Biglari et al. paper shows some interesting tricks for manipulating the plasma dispersion function Z (used also in Beer's thesis).

More ITG References (2)

- Kotschenreuther, Dorland, Beer, Hammett, PoP 1995,
  Presents the "IFS-PPPL" transport model, based on nonlinear gyrofluid ITG simulations and linear gyrokinetic simulations for a more accurate critical gradient. The first transport model comprehensive enough to successfully predict the temperature profiles in the core region of tokamaks over a wide range of parameters, including explaining the improved confinement of "supershots" and H-modes relative to L-modes. Also emphasized the importance of marginal stability effects that make core temperature profiles sensitive to edge temperature boundary conditions.

- Jenko, Dorland, Hammett, PoP 2001
  improved, fairly accurate critical gradient for ETG/ITG instabilities, fit to a large number of linear numerical gyrokinetic simulations (and recovers previous analytic results in various limits)

- "Comparisons and Physics Basis of Tokamak Transport Models and Turbulence Simulations", Dimits et al, PoP 2000
  Detailed cross-code comparisons of gyrofluid and full gyrokinetic codes for ITG turbulence (the "cyclone" case here is an oft-used benchmark test). Demonstrated that gyrofluid codes had too much damping of zonal flows and missed the "Dimits" nonlinear shift in the effective critical gradient. (These errors were not large enough to significantly affect previous predictions using gyrofluid-based models about the performance of the 1996 ITER design.) Later improvements to gyrofluid closures reduce the discrepancies.

More ITG References (3)

- Jenko & Dorland et al, PoP 2000, Dorland & Jenko et al. PRL 2000
  discovery that ETG turbulence is much stronger than expected from simple scaling from ITG turbulence, because of the important difference between the adiabatic species response to zonal flows.

- Jenko & Dorland, PRL 2002 http://prl.aps.org/abstract/PRL/v89/i22/e225001
  interesting explanation of the differences between ITG & ETG nonlinear saturation levels in various regimes based on secondary instability analysis, relative importance of Rogers (perpendicular/zonal flow) vs. Cowley (parallel flow) secondary instabilities.

- "Anomalous Transport Scaling in the DIII-D Tokamak Matched by Supercomputer Simulation", Candy & Waltz, PRL 2003, https://fusion.gat.com/THEORY/images/e/e7/Candy-PRL03.pdf
  One of the first comprehensive simulations by the GYRO code, similar to the Kotschenreuther-Dorland continuum gyrokinetic turbulence code, but extended from the local limit to consider non-local/global effects that can break gyro-Bohm scaling.

Gyrokinetic Turbulence Code References

- Below are 3 widely-used gyrokinetic codes for comprehensive 5-D plasma turbulence simulations. These 3 codes use "continuum" methods with a grid in phase-space, instead of the random sampling of Particle-in-Cell (PIC) algorithms. These 3 codes are relatively comprehensive, handling fully electromagnetic fluctuations with a kinetic treatment of electrons and multiple ion species, collision operators, and general non-circular tokamak geometries. They are actively being used to compare with experiments and to understand the underlying physics of the turbulence.

  – GS2 (Kotschenreuther & Dorland, IFS/Texas & Maryland) the first fully electromagnetic nonlinear gyrokinetic code, optimized for the small ρ_* thin-annulus / flux-tube local limit, and can also handle stellarators: http://gyrokinetics.sourceforge.net/

  – GENE (Jenko et al., Garching) similar to GS2 originally, extended to non-local/global effects like GYRO, and for stellarators: http://www.ipp.mpg.de/~fsj/gene

  – GYRO (Candy and Waltz et al., General Atomics), inspired by GS2, but extended to non-local global effects that can break gyro-Bohm scaling: http://fusion.gat.com/theory/Gyro

  – There are several PIC codes that have also been used to study aspects of tokamak turbulence with various levels of approximation, including GEM, ORB5, GTS, GTC, XGC, …

Further Reading for Newcomers to Plasmas

- The textbook by Goldston and Rutherford, "Introduction to Plasma Physics", is aimed at an advanced undergraduate level, and is a good place to start for those looking for a systematic treatment of plasma physics. In the back are several chapters that deal with the types of instabilities that drive small-scale turbulence in tokamaks (including the ITG instability and drift wave instabilities in simple slab geometry).
- Wesson's text book, "Tokamaks", is a nice compendium, and has sections on simple models of plasma turbulence and transport.
- Someday I should write up a more systematic description of the ideas I discuss here about simple pictures of ITG turbulence mechanisms, subtle effects of critical gradients, and a survey of ways to reduce turbulence.
- John Krommes, "The Gyrokinetic Description of Microturbulence in Magnetized Plasmas", Ann. Rev. of Fluid Mechanics 44, 175 (2012), http://dx.doi.org/10.1146/annurev-fluid-120710-101223 This is a survey of very interesting new results in tokamak turbulence. It discusses some cutting-edge research that is quite complicated, but tries to do so in way that gets some of the main ideas across to a broad audience of scientists outside of fusion research.

- Ph.D. Dissertations are a good place to look for beginners in a field, because they often contain useful tutorials or pointers to good references in the beginning sections. On the topic of tokamak turbulence, I would suggest dissertations by my recent students Luc Peterson and Jessica Baumgaertel, which are linked to at http://w3.pppl.gov/~hammett/papers/. (Granstedt's thesis is also very good, but has less intro material on turbulence.)
- My second Ph.D. student's thesis (Mike Beer 1995) has a good tutorial on the toroidal ITG mode: http://w3.pppl.gov/~hammett/collaborators/mbeer/afs/thesis.html
  Presents a tutorial on fundamentals and physical pictures of ITG mode, and the first comprehensive 3D gyrofluid simulations (gyrofluid equations include models of FLR & kinetic effects like Landau damping) of ITG and TEM turbulence in realistic toroidal geometry. Documents the important role of turbulence-generated zonal flows in saturating toroidal ITG turbulence, and the major reduction of ITG turbulence by using a proper adiabatic electron response that does not respond to zonal electric fields with $E_{||}=0$ (also shown in slab limit in Dorland's earlier thesis).