Low-beta electromagnetic plasma turbulence driven by electron-temperature gradient

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Motivation

- Understanding (turbulent) heat transport in magnetically confined plasmas is crucial to the design of successful tokamak experiments
- Therefore, we need to determine the turbulent state of the plasma at saturation
- We shall focus on turbulence driven by the electron-temperature gradient (ETG) instability
- Most theory is done electostatically; that is, under the assumption that magnetic field lines are not frozen into the electron velocity, and so electrons are free to stream across said field lines without deforming them
- Developing picture for the turbulent state in the electromagnetic regime is a natural and desirable extension of the typical ETG picture

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Local (slab) approximation

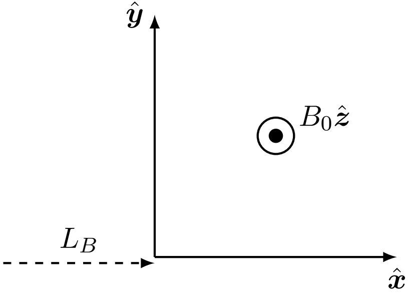

Figure 1: A representation of the magnetic geometry adopted. The domain is located a distance L_B from the axis, and $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are the 'radial' and 'poloidal' directions respectively. The equilibrium magnetic field B_0 is orientated along $\hat{\boldsymbol{z}}$.

$$L_B^{-1} = -\frac{1}{B_0}\frac{\mathrm{d}B_0}{\mathrm{d}x}, \quad L_{n_s}^{-1} = -\frac{1}{n_s}\frac{\mathrm{d}n_s}{\mathrm{d}x}, \quad L_{T_s}^{-1} = -\frac{1}{T_s}\frac{\mathrm{d}T_s}{\mathrm{d}x}.$$

Low-beta ordering

For characteristic frequencies ω and wavenumbers $k_\parallel$ and $k_\perp$ parallel and perpendicular to the total magnetic field $\boldsymbol{B}$, we adopt the ordering

$$\omega \sim k_\perp v_E \sim \omega_{ds} \sim k_\parallel v_{\mathrm{th}e} \sim \omega_{*s} \sim \omega_{\mathrm{KAW}} \sim \nu_{ee} \sim \nu_{ei}.$$

In particular, such an ordering implies both that the electron beta is small

$$\beta_e \sim \frac{Z m_e}{m_i} \ll 1, \quad \beta_e = \frac{8\pi n_e T_e}{B_0^2},$$

and that the perturbations are small-amplitude and highly anisotropic:

$$\frac{k_\parallel}{k_\perp} \sim \epsilon\sqrt{\beta_e}, \quad \epsilon = \frac{d_e}{L_{T_s}} \ll 1, \quad d_e = \frac{c}{\omega_{\mathrm{pe}}}.$$

In general, our system is captured by the fields ϕ, $A_\parallel$, δf_e, (g_i).

Gyrokinetics

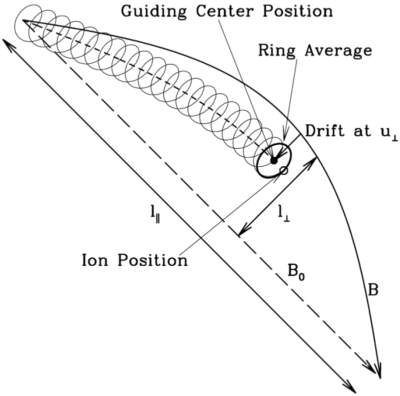

Figure 2: Within magnetic confinement fusion devices, particles perform gyro-motion around the local magnetic field. Gyrokinetics averages over this fast timescale, approximating the particles as rings of charge (diagram from Howes et al. (2006)).

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Electrostatic ETG Limit

We consider the subsidiary ordering

$$\frac{\omega_{de}}{\omega_{*e}} \ll \frac{k_\parallel v_{\text{the}}}{\omega_{*e}} \ll \frac{\omega}{\omega_{*e}} \ll 1 \ll k_\perp d_e,$$

under which our equations reduce to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\tau}^{-1}\varphi = \nabla_\parallel u_{\parallel e} - 2\frac{\rho_e v_{\text{the}}}{2L_B}\frac{\partial}{\partial y}\frac{\delta T_e}{T_e},$$
$$\frac{\mathrm{d}u_{\parallel e}}{\mathrm{d}t} = -\frac{v_{\text{the}}^2}{2}\nabla_\parallel \frac{\delta T_e}{T_e}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\text{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y},$$

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where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_E \cdot \nabla_\perp = \frac{\partial}{\partial t} + \frac{1}{2}\rho_e v_{\rm the}\{\varphi, ...\},$$

$$\nabla_\parallel = \boldsymbol{b}\cdot\nabla = \frac{\partial}{\partial z} + \frac{\delta \boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp = \frac{\partial}{\partial z} - \rho_e\{\mathcal{A}, ...\},$$

and we have defined the normalised variables $\varphi = e\phi/T_e$ and $\mathcal{A} = A_\parallel/\rho_e B_0$.

Electrostatic ETG Limit

We consider the subsidiary ordering

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$$\frac{\mathrm{d}u_{\parallel e}}{\mathrm{d}t} = -\frac{v_{\mathrm{th}e}^2}{2}\nabla_\parallel \frac{\delta T_e}{T_e}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\mathrm{th}e}}{2L_{T_e}}\frac{\partial \varphi}{\partial y},$$

In the absence of magnetic drifts ($L_B^{-1} \to 0$) we recover the familiar slab ETG result:

$$\omega = \mathrm{sgn}(k_y)\left(-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)\left(\frac{k_\parallel^2 v_{\mathrm{th}e}^2 |\omega_{*e}|\bar{\tau}}{2}\right)^{1/3}.$$

Electrostatic ETG Limit

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$$\frac{\mathrm{d}u_{\parallel e}}{\mathrm{d}t} = -\frac{v_{\rm the}^2}{2}\nabla_\parallel\frac{\delta T_e}{T_e}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\rm the}}{2L_{T_e}}\frac{\partial\varphi}{\partial y},$$

In the 2-D limit ($k_\parallel \to 0$), we obtain the familiar curvature-ETG mode

$$\omega^2 = -\omega_{*e}\omega_{de}\bar{\tau} = -\frac{(k_y\rho_e v_{\rm the})^2\bar{\tau}}{L_{T_e}L_B}$$

Electrostatic ETG Limit

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The conventional (slab) ETG instability exists as long as $k_\perp d_e$ is large, as it requires electrons to flow across field lines. However, since the curvature-ETG mode arises from the interchange of magnetic field lines, it will drive instabilities regardless of the perpendicular scale...

Electromagnetic KAW Limit

We considering the subsidiary ordering

$$\frac{\omega_{de}}{\omega_{*e}} \ll \frac{\omega}{\omega_{*e}} \sim k_\perp d_e \ll \frac{k_\| v_{\mathrm{the}}}{\omega_{*e}} \sim 1,$$

under which our equations reduce to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\tau}^{-1}\varphi = d_e^2 \nabla_\| \nabla_\perp^2 \mathcal{A} - 2\frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\frac{\delta T_e}{T_e},$$

$$\nabla_\| \frac{\delta T_e}{T_e} = \frac{\rho_e}{L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

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$$\nabla_\parallel \frac{\delta T_e}{T_e} = \frac{\rho_e}{L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\text{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

Linearising and Fourier transforming, we find the dispersion relation

$$\omega^2 = k_\parallel^2 v_{\text{the}}^2 k_\perp^2 d_e^2 \frac{\bar{\tau}}{2} - \omega_{*e}\omega_{de}\bar{\tau}.$$

Electromagnetic KAW Limit

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$$\nabla_\parallel \frac{\delta T_e}{T_e} = \frac{\rho_e}{L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

These hybrid KAW-curvature-ETG like modes are unstable for certain values of $k_\parallel$:

$$k_\parallel^2 \leqslant \frac{\beta_e}{2L_{T_e}R}\left(\frac{k_y}{k_\perp}\right)^2.$$

For $k_\parallel = 0$, we simply re-obtain the curvature-ETG mode.

Electromagnetic KAW Limit

We considering the subsidiary ordering

$$\frac{\omega_{de}}{\omega_{*e}} \ll \frac{\omega}{\omega_{*e}} \sim k_\perp d_e \ll \frac{k_\parallel v_{\mathrm{the}}}{\omega_{*e}} \sim 1,$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\tau}^{-1}\varphi = d_e^2 \nabla_\parallel \nabla_\perp^2 \mathcal{A} - 2\frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\frac{\delta T_e}{T_e},$$
$$\nabla_\parallel \frac{\delta T_e}{T_e} = \frac{\rho_e}{L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_e}{T_e} = -\frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

It thus appears that the curvature-ETG instability persists as a source of energy injection above the d_e scale. How does this affect our picture of the turbulent state of our plasma?

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Free energy and critical balance

Our system nonlinearly conserves the free energy

$$\frac{W}{n_e T_e} = \int \frac{\mathrm{d}^3 \boldsymbol{r}}{V} \left[\frac{\varphi \bar{\tau}^{-1} \varphi}{2} + |d_e \nabla_\perp \mathcal{A}|^2 + \frac{1}{2} \frac{\delta n_e^2}{n_e^2} + \frac{u_{\parallel e}^2}{v_{\mathrm{the}}^2} + \frac{1}{4} \frac{\delta T_{\parallel e}^2}{T_e^2} + \dots \right],$$

which is injected through equilibrium gradients and dissipated by collisions, leading to cascade of energy from large to small scales.

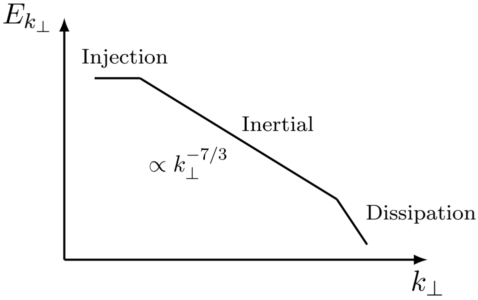

Figure 3: The typical Kolmogorov picture of turbulence.

Free energy and critical balance

Our system nonlinearly conserves the free energy

$$\frac{W}{n_e T_e} = \int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\frac{\varphi\bar{\tau}^{-1}\varphi}{2} + |d_e \nabla_\perp \mathcal{A}|^2 + \frac{1}{2}\frac{\delta n_e^2}{n_e^2} + \frac{u_{\|e}^2}{v_{\mathrm{the}}^2} + \frac{1}{4}\frac{\delta T_{\|e}^2}{T_e^2} + \dots\right],$$

which is injected through equilibrium gradients and dissipated by collisions, leading to cascade of energy from large to small scales. Assume:

- *Critical balance*: that the characteristic time for propagation along the field lines is comparable to the nonlinear advection rate t_{nl}^{-1} at each scale $k_\perp^{-1}$.
- That the perturbations are roughly isotropic in the perpendicular plane, so $k_x \sim k_y \sim k_\perp$.
- That there is a constant flux of energy in the inertial range:

$$\varepsilon_W \sim \frac{1}{n_e T_e}\frac{\mathrm{d}W}{\mathrm{d}t} \sim \frac{\bar{\tau}^{-1}\varphi^2}{t_{\mathrm{nl}}} = \text{constant}.$$

Turbulent heat fluxes

We have contributions to the turbulent heat flux arising from the conventional ETG instability and the curvature-ETG instability:

$$\boxed{(Q_{\rm turb})^{\rm ETG} \sim n_e T_e \frac{\rho_e^2 d_e}{L_{T_e}^2}\Omega_e \quad \text{vs.} \quad (Q_{\rm turb})^{\rm cETG} \sim n_e T_e \frac{\rho_e^3 L_\parallel}{L_{T_e} L_B^2}\Omega_e.}$$

- The transport that results from the curvature-ETG is less ‘stiff’ than that resulting from the ETG.
- Role of the parallel temperature perturbation is to maintain pressure balance along the perturbed field line, rather than drive parallel electron velocity.
- This means that the ETG only enters into the dispersion relation through the curvature term.

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Summary and future work

- We considered the turbulent state of a low-beta, magnetised plasma allowing electromagnetic perturbations and a variation of the equilibrium magnetic field
- In certain analytical limits, it was shown that the system supported two primary instabilities: the conventional electrostatic ETG instability, and KAWs unstable to the curvature-ETG instability.
- Below the d_e scale, the resultant turbulent heat flux scales as $L_{T_e}^{-2}$, while above the d_e scale it appears less 'stiff', scaling as $L_{T_e}^{-1}$.
- Simulations of a reduced version of the full kinetic system are currently being implemented in order to verify these analytical estimates.

Backup slides (1)

Gyrokinetic set-up:

$$\delta f_s(\boldsymbol{r}, \boldsymbol{v}, t) = -\frac{q_s \phi(\boldsymbol{r}, t)}{T_s} f_{0s}(x, \boldsymbol{v}) + h_s(\boldsymbol{R}_s, v_\perp, v_\parallel, t),$$

$$\frac{\partial}{\partial t}\left(h_s - \frac{q_s \langle \chi \rangle_{\boldsymbol{R}_s}}{T_s} f_{0s}\right) + v_\parallel \frac{\partial h_s}{\partial z} + \left(\boldsymbol{v}_{ds} + \langle v_\chi \rangle_{\boldsymbol{R}_s}\right) \cdot \nabla_\perp \left(h_s + f_{0s}\right) = C[h_s],$$

$$\boldsymbol{v}_\chi = \frac{c}{B_0} \hat{\boldsymbol{z}} \times \nabla_\perp \chi, \quad \chi = \phi - \frac{v_\parallel A_\parallel}{c} - \frac{\boldsymbol{v}_\perp \cdot \boldsymbol{A}}{c}$$

$$\boldsymbol{v}_{ds} = \frac{\boldsymbol{b}}{\Omega_s} \times \left[v_\parallel^2 \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} + \frac{1}{2} v_\perp^2 \boldsymbol{\nabla} \log B\right].$$

Backup slides (2)

Ordering of lengthscales:

$$k_\perp \rho_i \sim \sqrt{\frac{\tau}{Z}}, \quad k_\perp \rho_e \sim \sqrt{\beta_e}, \quad k_\perp d_e \sim 1,$$

$$k_\parallel L_{n_s} \sim k_\parallel L_{T_s} \sim k_\parallel R \sim \sqrt{\beta_e}, \quad \frac{k_\parallel}{k_\perp} \sim \epsilon\sqrt{\beta_e}.$$

Ordering of timescales:

$$\frac{\omega}{\Omega_e} \sim \epsilon\beta_e, \quad \frac{\omega}{\Omega_i} \sim \epsilon.$$

Ordering of perturbations of density, pressure and electromagnetic fields:

$$\frac{\delta n_e}{n_e} \sim \frac{\delta n_i}{n_i} \sim \frac{\delta p_e}{p_e} \sim \frac{\delta p_i}{p_i} \sim \frac{e\phi}{T_e} \sim \epsilon, \quad \frac{\delta B_\perp}{B_0} \sim \epsilon\sqrt{\beta_e}, \quad \frac{\delta B_\parallel}{B_0} \sim \epsilon\beta_e.$$

Backup slides (3)

Non-adiabatic response of the ions:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{v}_{di} \cdot \nabla_\perp\right) g_i + \frac{c}{B_0} \left\{ \langle \phi \rangle_{\boldsymbol{R}_i} - \phi, g_i \right\} + \langle \boldsymbol{v}_E \rangle_{\boldsymbol{R}_i} \cdot \nabla_\perp f_{0i}$$
$$= C\left[g_i + \frac{q_i \langle \phi \rangle_{R_i}}{T_i} f_{0i} \right],$$

where

$$g_i = h_i - \frac{q_i \langle \phi \rangle_{R_i}}{T_i} f_{0i}.$$

Quasineutrality and parallel Ampere's law:

$$\frac{\delta n_e}{n_e} = -\bar{\tau}^{-1}\varphi + \frac{1}{n_i}\int \mathrm{d}^3\boldsymbol{v}\ \langle g_i \rangle_{\boldsymbol{r}}\,, \quad \frac{u_{\parallel e}}{v_{\mathrm{the}}} = d_e^2 \nabla_\perp^2 \mathcal{A}.$$

where we have defined the normalised variables $\varphi = e\phi/T_e$ and $\mathcal{A} = A_\parallel/\rho_e B_0$.

Backup slides (4)

The electrons are drift kinetic, since $k_\perp \rho_e \sim \sqrt{\beta_e} \ll 1$, vis.:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_{de} \cdot \nabla_\perp\right) \delta f_e = -\boldsymbol{v}_\chi \cdot \nabla_\perp f_{0e} - \frac{v_\parallel e E_\parallel}{T_e} + C[\delta f_e].$$

We choose to expand δf_e in Laguerre-Hermite moments $g_{\ell,m}$:

$$g_{\ell,m}(\boldsymbol{r}, t) = \frac{1}{n_e} \int \mathrm{d}^3\boldsymbol{v}\, (-1)^\ell \frac{L_\ell(v_\perp^2/v_{\mathrm{th}e}^2) H_m(v_\parallel/v_{\mathrm{th}e})}{\sqrt{2^m m!}}\, \delta f_e,$$

$$\delta f_e(\boldsymbol{r}, v_\parallel, v_\perp^2, t) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} (-1)^\ell \frac{L_\ell(v_\perp^2/v_{\mathrm{th}e}^2) H_m(v_\parallel/v_{\mathrm{th}e}) f_{0e}}{\sqrt{2^m m!}}\, g_{\ell,m},$$

where

$$L_\ell = \text{Laguerre polynomials of order } \ell,$$

$$H_m = \text{Hermite polynomials of order } m.$$

Backup slides (5)

$$L_\ell(\mu) = \frac{e^\mu}{\ell!}\frac{\mathrm{d}^\ell}{d\mu^\ell}(e^{-\mu}\mu^\ell), \quad \int \mathrm{d}\mu\, L_\ell(\mu)L_{\ell'}(\mu)e^{-\mu} = \delta_{\ell\ell'},$$

$$H_m(\hat{v}) = (-1)^m e^{\hat{v}^2}\frac{\mathrm{d}^m}{\mathrm{d}\hat{v}^m}e^{-\hat{v}^2}, \frac{1}{\sqrt{\pi}}\int \mathrm{d}\hat{v} H_m(\hat{v})H_{m'}(\hat{v})e^{-\hat{v}^2} = 2^m m!\delta_{mm'},$$

$$\mu L_\ell = (2\ell+1)L_\ell - (\ell+1)L_{\ell+1} - \ell L_{\ell-1}, \quad \frac{\mathrm{d}L_\ell}{\mathrm{d}\mu} = \frac{\mathrm{d}L_{\ell-1}}{\mathrm{d}\mu} - L_{\ell-1},$$

$$\hat{v}H_m = \frac{1}{2}H_{m+1} + mH_{m-1}, \quad \frac{\mathrm{d}H_m}{\mathrm{d}\hat{v}} = 2mH_{m-1}.$$

Backup slides (6)

The Laguerre-Hermite transform allows us to express the electron gyrokinetic equation in terms of a series of 'fluid moments':

$$\frac{\mathrm{d}g_{\ell,m}}{\mathrm{d}t} + \frac{v_{\mathrm{the}}}{\sqrt{2}}\nabla_{\|}(\sqrt{m+1}\, g_{\ell,m+1} + \sqrt{m}\, g_{\ell,m-1}) \\ - C[g_{\ell,m}] + \omega_{de}[g_{\ell,m}] = (\omega_{*e})_{\ell,m}.$$

We have defined:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_E \cdot \nabla_\perp = \frac{\partial}{\partial t} + \frac{1}{2}\rho_e v_{\mathrm{the}}\{\varphi, ...\},$$

$$\nabla_{\|} = \boldsymbol{b}\cdot\nabla = \frac{\partial}{\partial z} + \frac{\delta\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp = \frac{\partial}{\partial z} - \rho_e\{\mathcal{A}, ...\},$$

which are the time derivative in the frame moving with the $E \times B$ flow, and the derivative along the exact, perturbed magnetic field line respectively.

Backup slides (7)

Electron-electron and electron-ion collisions:

$$C[g_{\ell,m}] = -(\nu_{ee} + \nu_{ei})(m + 2\ell)g_{\ell,m} + \nu_{ee}g_{0,1}\delta_{0,1} + \frac{1}{3}(\nu_{ee} + \nu_{ei})\left(\sqrt{2}g_{0,2} + 2g_{1,0}\right)\left(\sqrt{2}\delta_{0,2} + 2\delta_{1,0}\right).$$

Magnetic drifts:

$$\omega_{de}[g_{\ell,m}] = \frac{\rho_e v_{\text{the}}}{2L_B}\frac{\partial}{\partial y}\Big[\sqrt{(m+1)(m+2)}g_{\ell,m+2} + 2(m+\ell+1)g_{\ell,m} + \sqrt{m(m-1)}g_{\ell,m-2} + (\ell+1)g_{\ell+1,m} + \ell g_{\ell-1,m}\Big].$$

Energy injection:

$$(\omega_{*e})_{\ell,m} = -\frac{\rho_e v_{\text{the}}}{2L_{n_e}}\frac{\partial\varphi}{\partial y}\left[\delta_{0,0} + \eta_e\left(\delta_{1,0} + \frac{1}{\sqrt{2}}\delta_{0,2}\right)\right] + \frac{\sqrt{2}\rho_e v_{\text{the}}}{2L_{n_e}}\frac{\partial\mathcal{A}}{\partial y}\left[\delta_{0,1} + \eta_e\left(\delta_{0,1} + \delta_{1,1} + \sqrt{\frac{3}{2}}\delta_{0,3}\right)\right] + \frac{v_{\text{the}}}{\sqrt{2}}\left(\frac{2}{v_{\text{the}}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} + \frac{\partial\varphi}{\partial z}\right)\delta_{0,1} + \frac{\rho_e v_{\text{the}}}{2L_B}\frac{\partial\varphi}{\partial y}\left[\sqrt{2}\delta_{0,2} + \delta_{1,0} + 2\delta_{0,0}\right]$$

Backup slides (8)

Density moment $(\ell, m) = (0, 0)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta n_e}{n_e} + \nabla_\parallel u_{\parallel e} + \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(\frac{\delta T_{\parallel e}}{T_e} + \frac{\delta T_{\perp e}}{T_e} + 2\frac{\delta n_e}{n_e} - 2\varphi\right) = 0.$$

Parallel velocity moment $(\ell, m) = (0, 1)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{u_{\parallel e}}{v_{\mathrm{the}}} + \frac{v_{\mathrm{the}}}{2}\nabla_\parallel\left(\frac{\delta T_{\parallel e}}{T_e} + \frac{\delta n_e}{n_e}\right) + \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(4\frac{u_{\parallel e}}{v_{\mathrm{the}}} + \frac{\delta q_{\parallel e} + \delta q_{\perp e}}{n_e v_{\mathrm{the}} T_e}\right)$$
$$= \frac{\rho_e v_{\mathrm{the}}}{2L_{n_e}}\frac{\partial \mathcal{A}}{\partial y}(1 + \eta_e) + \left(\frac{\partial \mathcal{A}}{\partial t} + \frac{v_{\mathrm{the}}}{2}\nabla_\parallel\varphi\right) - \nu_{ei}\frac{u_{\parallel e}}{v_{\mathrm{the}}}.$$

Backup slides (9)

Parallel temperature moment $(\ell, m) = (0, 2)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_{\parallel e}}{T_e} + v_{\mathrm{the}}\nabla_\parallel\left(\frac{\delta q_{\parallel e}}{n_e v_{\mathrm{the}} T_e} + 2\frac{u_{\parallel e}}{v_{\mathrm{the}}}\right) + \frac{4}{3}(\nu_{ee} + \nu_{ei})\frac{\delta T_{\parallel e} - \delta T_{\perp e}}{T_e}$$
$$+ \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(6\frac{\delta T_{\parallel e}}{T_e} + 2\frac{\delta n_e}{n_e} - 2\varphi + 2\sqrt{6}g_{0,4} + \sqrt{2}g_{1,2}\right) = -\frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

Perpendicular temperature moment $(\ell, m) = (1, 0)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T_{\perp e}}{T_e} + v_{\mathrm{the}}\nabla_\parallel\frac{\delta q_{\perp e}}{n_e v_{\mathrm{the}} T_e} + \frac{2}{3}(\nu_{ee} + \nu_{ei})\frac{\delta T_{\perp e} - \delta T_{\parallel e}}{T_e}$$
$$+ \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(4\frac{\delta T_e}{T_e} + \frac{\delta n_e}{n_e} - \varphi + 2g_{2,0} + \sqrt{2}g_{1,2}\right) = -\frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \varphi}{\partial y}.$$

Backup slides (10)

Parallel heat flux moment $(\ell, m) = (0, 3)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta q_{\parallel e}}{n_e v_{\mathrm{the}} T_e} + v_{\mathrm{the}} \nabla_{\parallel} \left(\sqrt{6} g_{0,4} + \frac{3}{2}\frac{\delta T_{\parallel e}}{T_e} \right) + 3(\nu_{ee} + \nu_{ei}) \frac{\delta q_{\parallel e}}{n_e v_{\mathrm{the}} T_e}$$
$$+ \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(2\sqrt{15} g_{0,6} + 8\frac{\delta q_{\parallel e}}{n_e v_{\mathrm{the}} T_e} + 6\frac{u_{\parallel e}}{v_{\mathrm{the}}} + \sqrt{3} g_{1,3} \right) = \frac{3\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}.$$

Perpendicular heat flux moment $(\ell, m) = (1, 1)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta q_{\perp e}}{n_e v_{\mathrm{the}} T_e} + v_{\mathrm{the}} \nabla_{\parallel} \left(\frac{1}{\sqrt{2}} g_{1,2} + \frac{1}{2}\frac{\delta T_{\perp e}}{T_e} \right) + 3(\nu_{ee} + \nu_{ei}) \frac{\delta q_{\perp e}}{n_e v_{\mathrm{the}} T_e}$$
$$+ \frac{\rho_e v_{\mathrm{the}}}{2L_B}\frac{\partial}{\partial y}\left(\sqrt{3} g_{1,3} + 6\frac{\delta q_{\perp e}}{n_e v_{\mathrm{the}} T_e} + \frac{u_{\parallel e}}{v_{\mathrm{the}}} + \sqrt{2} g_{2,1} \right) = \frac{\rho_e v_{\mathrm{the}}}{2L_{T_e}}\frac{\partial \mathcal{A}}{\partial y}.$$

Backup slides (11)

(ETG turbulence) In the inertial range, we have the critical balance:

$$t_{\rm nl}^{-1} \sim \rho_e v_{\rm the} k_\perp^2 \varphi \sim \Omega_e^{2/3} \varepsilon_W^{1/3} \bar{\tau}^{1/3} (k_\perp \rho_e)^{4/3} \sim k_\parallel v_{\rm the}.$$

At the outer scale, the maximal growth rate is comparable to the streaming rate,

$$k_\parallel^o v_{\rm the} \sim \gamma_{\rm ETG}^o \sim \omega_{*e}^o \eta_e \quad \Rightarrow \quad k_\parallel^o L_{T_e} \sim k_y^o \rho_e \sim \frac{\Omega_e}{\varepsilon_W} \left(\frac{\rho_e}{L_{T_e}} \right)^3 .$$

The ETG instability is stabilised by magnetic tension at $k_\perp d_e \sim 1$, meaning that the outer scale is pinned at $k_y^o d_e \sim 1$. The turbulent rate of energy injection and heat flux are then:

$$\varepsilon_W \sim \frac{\rho_e^2 d_e}{L_{T_e}^3} \Omega_e \quad \Rightarrow \quad \boxed{(Q_{\rm turb})^{\rm ETG} \sim n_e T_e \frac{\rho_e^2 d_e}{L_{T_e}^2} \Omega_e.}$$

Backup slides (12)

(Curvature-ETG turbulence) We argue that realistic turbulence set up by the curvature-ETG will sit in kinetic regime, where $\omega \sim k_\parallel v_{\mathrm{th}e}$.

- Assuming that the turbulence exists at scales $k_\perp \rho_i \gtrsim 1$, we have (at the outer scale):

$$k_\parallel^o (L_{T_e} L_B)^{1/2} \sim k_y^o \rho_e \sim \frac{\Omega_e}{\varepsilon_W} \frac{\rho_e^3}{(L_{T_e} L_B)^{3/2}}.$$

- Maximum growth rate occurs for $k_\parallel = 0$; in a finite system, this is limited by the parallel system size $L_\parallel \sim q L_B$.
- It follows that the turbulent rate of energy injection and heat flux are:

$$\varepsilon_W \sim \frac{\rho_e^3 L_\parallel}{L_{T_e}^2 L_B^2} \Omega_e \quad \Rightarrow \quad \boxed{(Q_{\mathrm{turb}})^{\mathrm{cETG}} \sim n_e T_e \frac{\rho_e^3 L_\parallel}{L_{T_e} L_B^2} \Omega_e.}$$

Backup slides (13)