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Geometry and Kinetics of Astrophysical Plasmas:

A gyrokinetic approach

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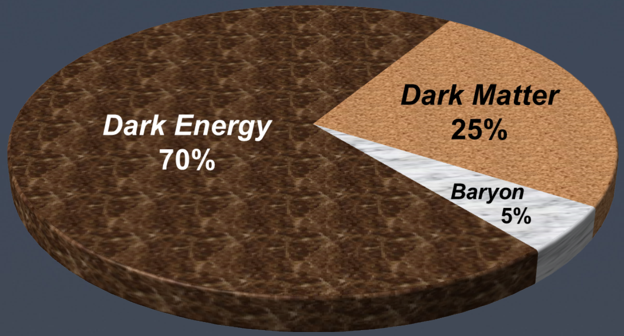
PPPL Graduate Summer School - 2019

Plasma is ubiquitous in the Universe

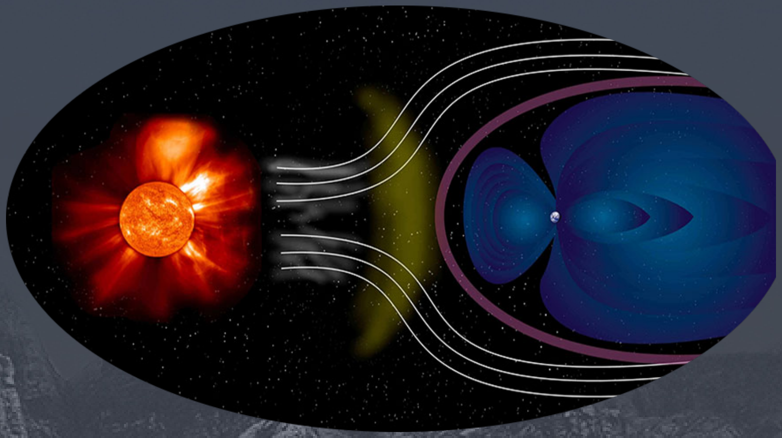


> 99% is in the plasma state

Examples of plasma in the universe

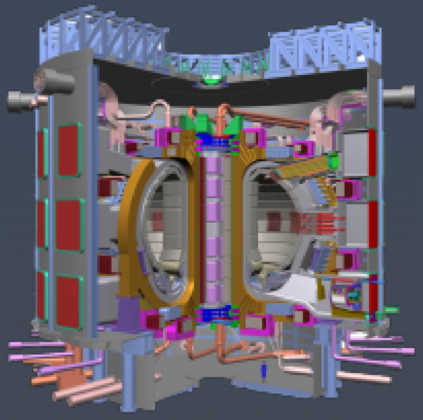
Accretion disks

Galaxy formation



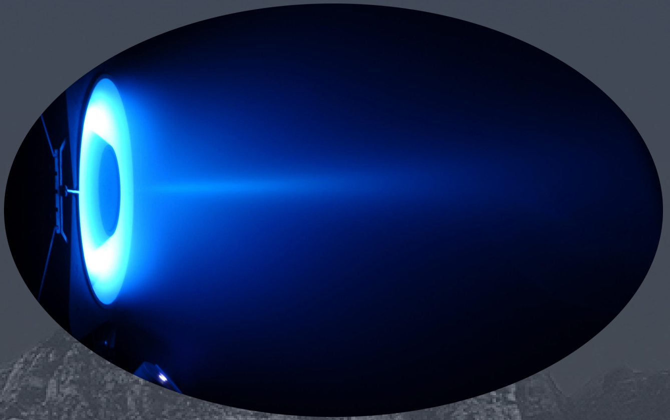
Solar wind

Examples of plasma in laboratory



Thermonuclear controlled fusion

Thin films in semiconductors



Hall thruster

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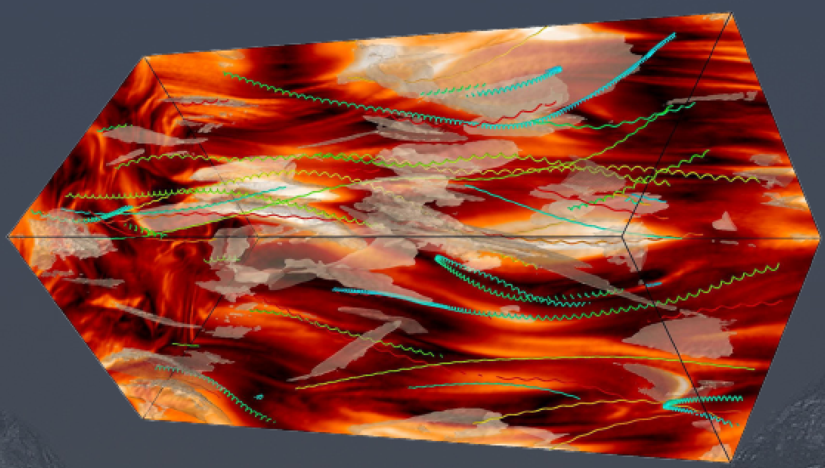
Magnetohydrodynamics

Kinetic theory

Turbulence plays a key role in understanding the dynamics of plasmas

Chaotic (yet deterministic) changes in various parameters → Turbulent behavior

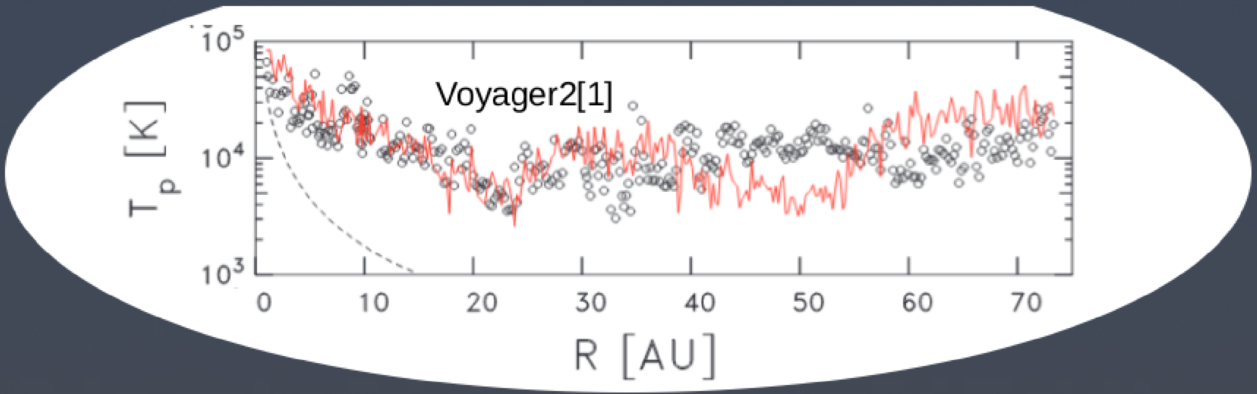


Supercomputer simulation of the motion of energetic particles, i.e. cosmic rays, in astrophysical plasma turbulence. (Graphic: Daniel Grošelj, IPP)

- Energy dissipation
- Particles transport and acceleration
- Generation of Magnetic fields → Dynamo
- ...

“The nature of the Alfvénic turbulence in the solar wind remains a major unsolved mystery”

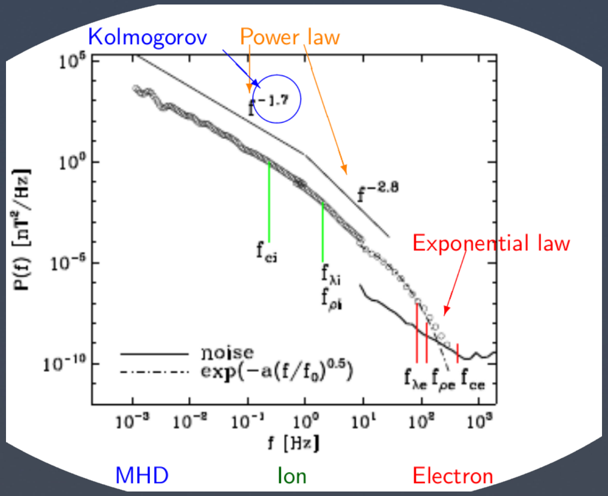
“Melvyn L. Goldstein, Major Unsolved Problems in Space Plasma Physics”



Adiabatic approximations fails → Turbulent heating

[1]The Astrophysical Journal, 638:508–517, 2006 February 10

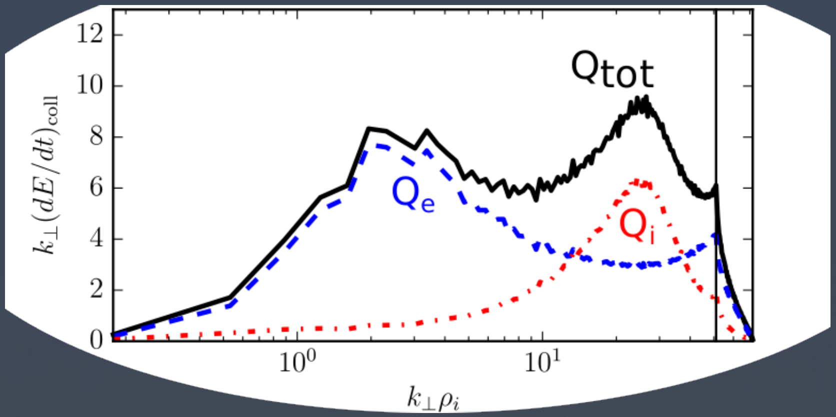
Magnetic spectrum in the solar wind



Richardson2003

+

Distribution of collisional dissipation[2]



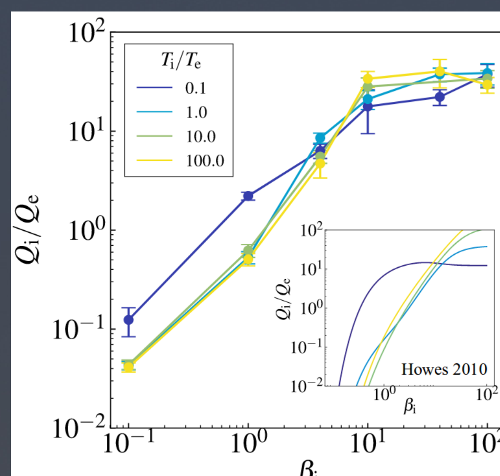
Electron heating at ion scale → Landau damping (Kinetic effect)

Electron kinetics is significant in astrophysical environment

[2]Phys. Rev. Lett. 115, 025003

Beyond solar wind...

- “ Within this scenario, the relative amount of electron heating in the low-β_i, central region of the disk turns out to be crucial to enable a detectable jet. ” [3]



→ Thermal disequilibration generated by imbalance on energy partition dependent on plasma beta.

[3] PNAS January 15, 2019 116 (3) 771-776
[4] Mon Not R Astron Soc 478:5209–5229

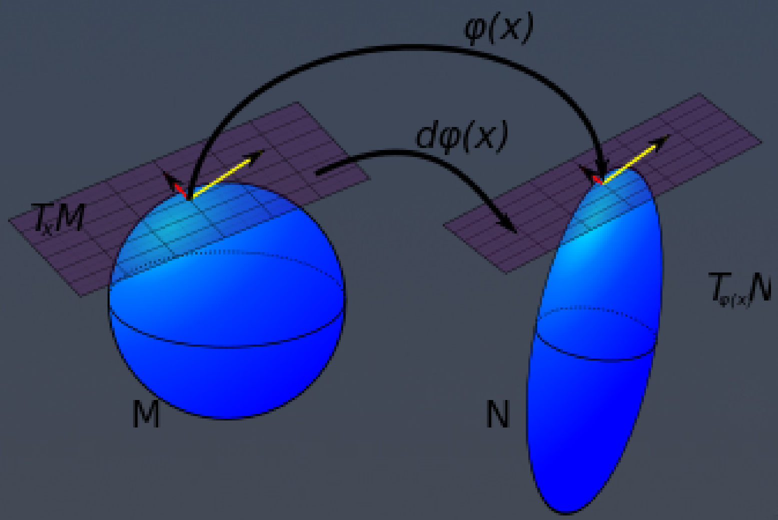
Geometrical approach to theoretical physics

- Geometry has the property of formalizing apparently independent physical systems into a consistent theoretical framework.

“Einstein went further; he wished to comprehend this single unified force - assuming that it existed -as a geometrical property of the space-time manifold we live in.” Abdu Salam in *Einstein's Last Dream: The Space -Time Unification of Fundamental Forces.*

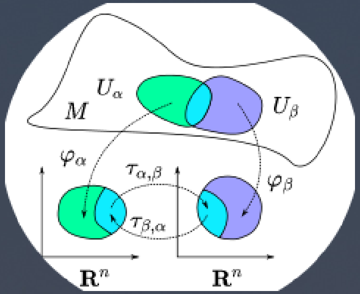
- In the context of plasma physics, we have a consistent model with no *ad hoc* elements, Hamiltonian structures preserved up to numerical implementation (i.e. energy conservation), and a more coherent* particle-fields interaction.

GK Diff. Geo. 101



- Analysis of geometric properties of differentiable manifolds.
- Differential forms as coordinate free objects (functions of vector fields on M) lying on the manifold and defining integrands over curves.
- Exterior derivatives as extension of differentiability of n-dimensional differential forms.

- In theoretical physics, mathematical mappings are interpreted as coordinate transformations



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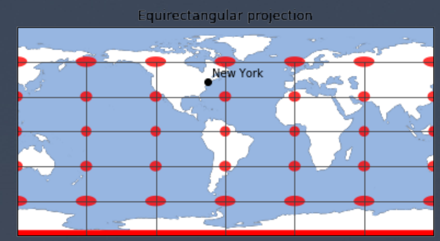
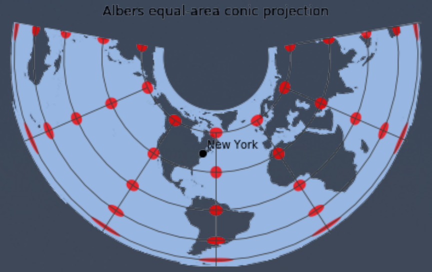
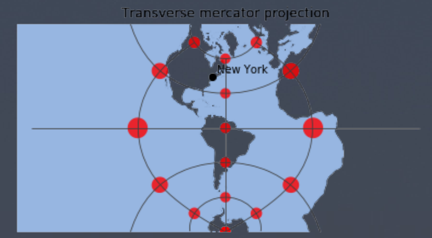
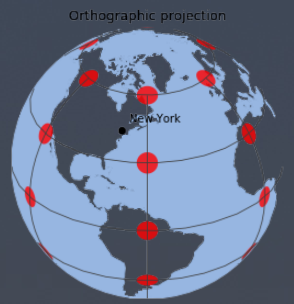








- Second external derivative keeps dynamics intact → Gauge

$$\nabla \times \nabla \phi = 0$$

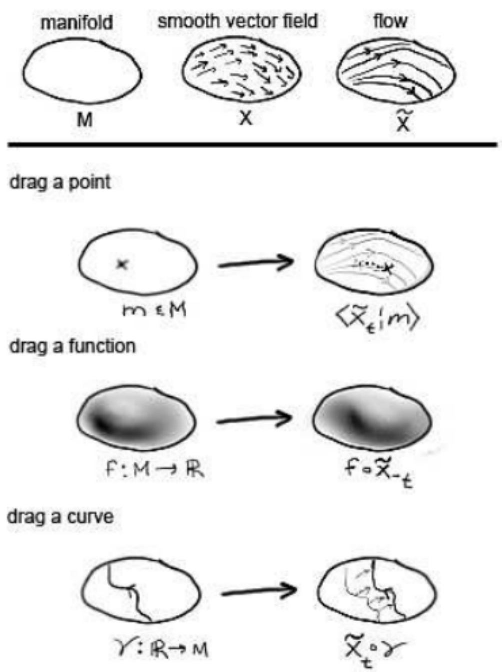
→

$$\nabla = (dx)\frac{\partial}{\partial x} + (dy)\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}$$

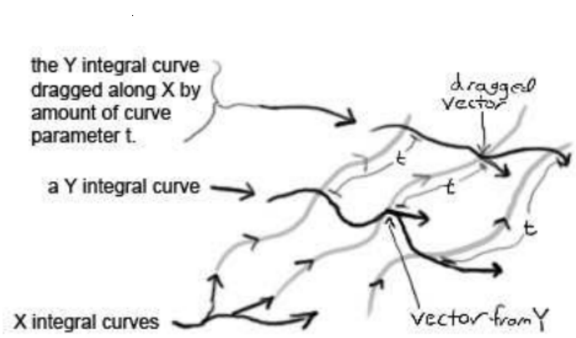
→

$$\nabla \times \nabla f = \frac{\partial^2 f}{\partial x \partial y} dx \times dy + \frac{\partial^2 f}{\partial y \partial x} dy \times dx + \ldots$$

Lie derivatives



Ebrahim(UCSB 2010)



$$(\mathcal{L}_Y T)_p = \frac{d}{dt}\bigg|_{t=0} \left((\varphi_{-t})_* T_{\varphi_t(p)}\right) = \frac{d}{dt}\bigg|_{t=0} \left((\varphi_t)^* T_p\right)$$

Pushforward maps are used to push tangent vectors on M forward to tangent vectors on N.

“The effect of an infinitesimal coordinate transformation on any tensor T is that the new tensor equals the old tensor at the same coordinate point, plus the Lie derivative of the tensor.”

"Gravitaion and Cosmology" by S. Weinberg

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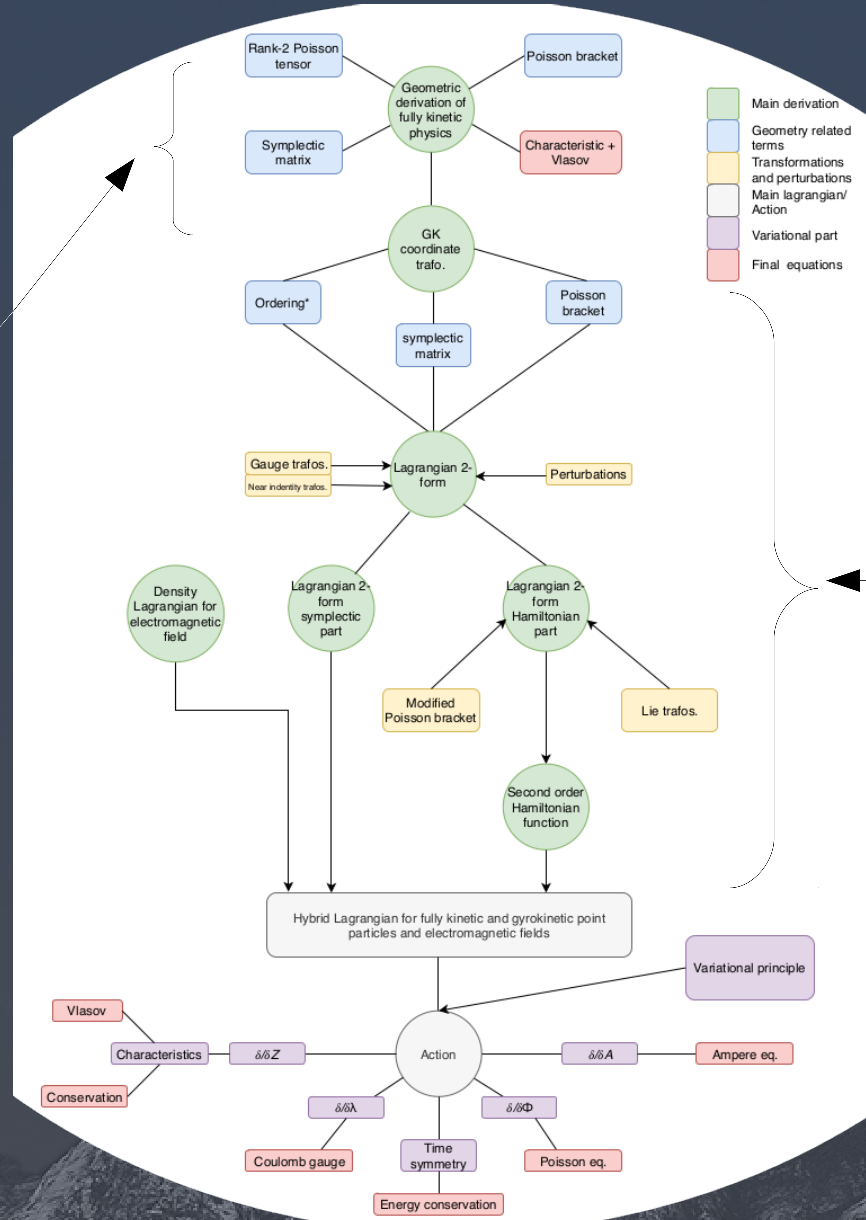
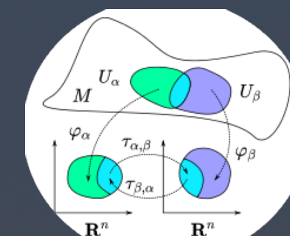
Fully Kinetic derivation





Vlasov + fields through variational principle

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Construction of symplectic matrix from the phase space Lagrangian

$$\omega_{\alpha\beta} = \epsilon_{ijk} B_k dx^i \wedge dx^j - m\delta_{ij} dx^i \wedge dv^j + m\delta_{ij} dv^i \wedge dx^j$$

Rank-2 Poisson tensor is constructed as the inverse of the symplectic tensor

$$\Pi^{\alpha\beta} = \omega^{-1}_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{m}\delta^{ij} \\ -\frac{1}{m}\delta^{ij} & \frac{1}{m^2}\epsilon^{ijk} B_k \end{pmatrix}$$

The Poisson Bracket defines a structure on the manifold that allow us to describe the dynamics of the system

$$\Pi(A, K) \equiv \{A, K\} = \sum_{\alpha\beta} \frac{\partial A}{\partial Z^\alpha} \Pi^{\alpha\beta} \frac{\partial K}{\partial Z^\beta} = \frac{1}{m}\left(\nabla A \frac{\partial K}{\partial v} - \frac{\partial A}{\partial v} \nabla K\right) + B\left(\frac{\partial A}{\partial v} \times \frac{\partial K}{\partial v}\right)$$

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Equations of motion are derived using the variational principle (or Poisson Bracket) with respect to the phase space

$$\dot{x}^I = \{x^I, H\} = \delta^{ij}\frac{\partial H}{\partial v^j} = \Pi^{x^i x^j}\frac{\partial H}{\partial x^j} + \Pi^{x^i v^j}\frac{\partial H}{\partial v^j} = v^I$$

$$\dot{v}^I = \{v^I, H\} = \Pi^{v^i x^i}\frac{\partial H}{\partial x^j} + \Pi^{v^i v^j}\frac{\partial H}{\partial v^j} = -\delta^{ij}\frac{\partial H}{\partial x^j} + \epsilon^{ijk}\frac{\partial H}{\partial v^i} = \frac{\partial \phi}{\partial x^i} + \epsilon^{ijk} v_j B_k$$

Liouville theorem → Vlasov equation (Astrophysical plasma → collisionless)

$$\frac{\partial F^I}{\partial t} + \{x^I, H^I\}\nabla F^I + \{v^I, H^I\}\partial_{v^I} F^I = 0.$$

Coordinate transformation

Ordering → $$\frac{(k_{\|}\rho_{th})e\phi_i}{T_i} = \epsilon_\delta \gg \epsilon_B = \frac{\rho_{th}}{|\nabla B/B|^{-1}}$$

Symplectic matrix from Lagrangian 2-form

$$\omega_{\alpha\beta} = \begin{pmatrix} \omega_{x^i x^j} & \omega_{x^i v^j} \\ \omega_{v^i x^j} & \omega_{v^i v^j} \end{pmatrix}$$

Euler-Lagrange equation → $$\omega_{\alpha\beta}\frac{dZ^\beta}{dt} = \frac{\partial H}{\partial Z^\alpha}$$

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Guiding center transformation

After introducing non uniformities in the magnetic field, one must separate the symplectic part of the Lagrangian in order to remove theta dependency.

$$\Gamma \rightarrow \Gamma_0 + \Gamma_1$$

Because we are working with 2-forms, the addition of gauge transformations in the Lagrangian does not change the dynamics of the system.

$$\sigma_1 = -\sum_{n=1} \frac{1}{n!} \frac{e}{\epsilon c} (\rho_0 \cdot \nabla)^{n-1} A \cdot \rho_0$$

A total of four gauge transformations are performed, together with one near identity coordinate transformation and one velocity shift. Theta dependency is removed from the symplectic part of the Lagrangian (up to ε_B).

The above transformations leave all the theta dependency up to the chosen ordering in the Hamiltonian part of the phase space Lagrangian.

Gyrocenter transformation

The final step consists of a Lie transformation on the perturbed non canonical Hamiltonian 0-form

$$\mathcal{H}_{gc}(\mathbb{Z}_{gc}) = e^{-\pounds_s}\mathcal{H}(\mathbb{Z})$$

Theta dependency is eliminated up to chosen ordering via the canonical Lie transformation on Hamiltonian part of the 2-form phase space Lagrangian. One needs to work with the modified Poisson Bracket in order to properly find the generators associated with the Lie Algebra of the above Lie transformations.

$$\{\mathcal{A},\mathcal{B}\}_{gy} = \epsilon^{-1}\left(\frac{\partial\mathcal{A}}{\partial\overline{\theta}}\frac{\partial\mathcal{B}}{\partial\overline{\mu}} - \frac{\partial\mathcal{B}}{\partial\overline{\mu}}\frac{\partial\mathcal{A}}{\partial\overline{\theta}}\right) + \\ +\frac{B^*}{B^*_\parallel}\left(\nabla^*\mathcal{A}\frac{\partial\mathcal{B}}{\partial\overline{v}_\parallel} - \frac{\partial\mathcal{A}}{\partial\overline{v}_\parallel}\nabla^*\mathcal{B}\right) - \\ -\epsilon\frac{\widehat{b}}{B^*_\parallel}(\nabla^*\mathcal{A}\times\nabla^*\mathcal{B})$$

$$H_{gy} = \frac{1}{2}mv^2_{\parallel gy} - \mu_{gy}B(\mathbf{X}_{gy}) - \varepsilon_\delta e\left\langle\psi_1\right\rangle - \varepsilon_\delta^2 e^2\left(\frac{1}{2mc^2}\left\langle|\mathbf{A}_1|^2\right\rangle - \frac{1}{2B(\mathbf{X}_{gy})}\partial_{\mu_{gy}}\left\langle\psi_1^2\right\rangle\right)$$

Full phase space Lagrangian constructed

$$\mathcal{L}^p_{gy} = \left(\frac{e}{\varepsilon c}\mathbf{A}_1 + m v_{\|gy}\hat{b}(\mathbf{X}_{gy})\right)\cdot \dot{X}_{gy} + \varepsilon\frac{mc}{e}\mu_{gy}\dot{\theta}_{gy} -$$

$$\frac{1}{2}m v^2_{\|gy} - \mu_{gy}B(\mathbf{X}_{gy}) - \varepsilon_\delta e\langle\psi_1\rangle - \varepsilon_\delta^2 e^2\left(\frac{1}{2mc^2}\langle|\mathbf{A}_1|^2\rangle - \frac{1}{2B(\mathbf{X}_{gy})}\partial_{\mu_{gy}}\left(\langle\psi_1^2\rangle - \langle\psi_1\rangle^2\right)\right)$$

$$\mathcal{L}^p_{fk} = \left(m_0\mathbf{v}_0 + \frac{e}{c}A(x_0,t)\right)\cdot\dot{x}_0 - \frac{1}{2}m_0|\mathbf{v}_0|^2 + e\phi(x_0,t)$$

$$\mathcal{L}^f = \frac{1}{8\pi}\int_v d^3x_0\left(\varepsilon_\delta^2|\nabla\phi_1(x_0)|^2 - |\nabla\times[\mathbf{A}_0(x_0) + \varepsilon_\delta\mathbf{A}_1(x_0,t)]|^2 + \varepsilon_\delta\frac{2}{c}\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t)\right)$$

The action becomes

$$\mathcal{S} = \int F(X_{gy}, v_{\|gy}, \mu_{gy}, t)\mathcal{L}^p_{gy}d\Omega dt + \int F(x_0, v_0, t)\mathcal{L}^p_{fk}d\Omega' dt + \int \mathcal{L}^f dt$$

Variational principle

Equations of Motion are derived as variations with respect to gyrokinetic phase space

$$\dot{\mathbf{X}}_{gy} = \{\mathbf{X}_{gy}, H_{gy}\} = \frac{B^*}{mB^*_{\parallel}} \frac{\partial H_{gy}}{\partial v_{\parallel gy}} + \frac{c\hat{b}}{eB^*_{\parallel}} \varepsilon \nabla^* H_{gy}$$

$$\dot{v}_{\parallel gy} = \{v_{\parallel gy}, H_{gy}\} = \frac{B^*}{mB^*_{\parallel}} \cdot \left(\nabla v_{\parallel gy} \frac{\partial H_{gy}}{\partial v_{\parallel gy}} - \nabla^* H_{gy} \right) - \frac{c\hat{b}}{eB^*_{\parallel}} \varepsilon \left(\nabla^* v_{\parallel gy} \times \nabla^* H_{gy} \right)$$

$$\dot{\mu}_{gy} = \{\mu_{gy}, H_{gy}\} = -\frac{e}{mc}\frac{1}{\varepsilon} \partial_{\theta_{gy}} H_{gy} + \frac{B^*}{mB^*_{\parallel}} \nabla \mu_{gy} - \frac{c\hat{b}}{eB^*_{\parallel}} \varepsilon \left(\nabla \mu_{gy} \times \nabla^* H_{gy} \right)$$

$$= -\frac{e}{mc}\frac{1}{\varepsilon} \partial_{\theta_{gy}} H_{gy} = 0$$

$$\dot{\theta}_{gy} = \frac{e}{mc}\frac{1}{\varepsilon} \partial_{\mu_{gy}} H_{gy} + \frac{\nabla^* \theta_{gy}}{\nabla^* X_{gy}} \cdot \frac{\partial \mathbf{X}_{gy}}{\partial t}$$

Conservation equation

We can also write down the gyrokinetic Vlasov equation in the conservation form. Taking in consideration the Jacobian of the gyrokinetic coordinate transformation , $\mathcal{J}(\Omega) = B_{\|}^{*}(X_{gy}, v_{\|gy}, \mu_{gy}, \theta_{gy})/m$ where Omega represents the total gyrokinetic phase space. The gyrocenter phase space conservation law can be derived as following

$$\frac{\partial}{\partial \Omega} \cdot [\mathcal{J}(\Omega)\{\Omega, H_{gy}(\Omega, t)\}] = 0$$

Considering we have

$$\mathcal{J}(\Omega)F(\Omega, t) = \int d^6\Omega_0 \mathcal{J}(\Omega_0)F(\Omega_0, t_0)\delta^6[\Omega - \Omega(\Omega_0, t_0; t)]$$

And

$$\frac{d\Omega}{dt} = \{\Omega, H_{gy}(\Omega, t)\}$$

We have finally

$$\frac{\partial}{\partial t}[\mathcal{J}(\Omega)F(\Omega, t)] + \frac{\partial}{\partial \Omega} \cdot [\mathcal{J}(\Omega)F(\Omega, t)\{\Omega, H_{gy}(\Omega, t)\}] = 0$$

$$\left[\frac{\partial}{\partial t} + \{\Omega, H_{gy}(\Omega, t)\}\frac{\partial}{\partial \Omega}\right] F(\Omega, t) = 0$$

Coulomb gauge

The gauge fixing is a mathematical maneuver used to remove redundant degrees of freedom in any field theory. In our case we have

$$\frac{\delta}{\delta\lambda}S = \frac{\delta}{\delta\lambda}\int\left\{\frac{1}{8\pi}\int_v d^3x_0\varepsilon_\delta\frac{2}{c}\lambda(x_0,t)\nabla\mathbf{A}_1(x_0,t)\right\}dt = 0$$

$$\frac{\delta}{\delta\lambda}\{\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t)\} = 0$$

$$\frac{\delta}{\delta\lambda}\lambda(x_0,t)\nabla\cdot\mathbf{A}_1(x_0,t) + \lambda(x_0,t)\nabla\cdot\left(\frac{\delta}{\delta\lambda}\mathbf{A}_1(x_0,t)\right) = 0$$

$$\nabla\cdot\mathbf{A}_1(x_0,t) = 0$$

which is the Columb gauge condition

Variation with respect to fields

- Electric potential → Poisson equation
  Using the functional derivative of the action:

$$\nabla_\perp^2 \phi_1 - \nabla_\perp \cdot \left(\frac{1}{8\pi} \frac{\rho_{th}^2}{\lambda_D} \nabla_\perp \phi_1 - 4 v_\| \frac{mc}{eB} \eta_0 \nabla_\perp A_{1\|} - \frac{6}{Be^2} \sqrt{\frac{2\mu_{gy} mc^2}{B}} \eta_0 \nabla_\perp A_{1\perp} \right) = \langle \eta \rangle$$

  - Electrostatic Poisson equation recovered in the absence of perturbed magnetic potential

- Magnetic potential → Split Ampere equation (good for high β plasma)

$$A_1 \rightarrow A_{1\|} + A_{1\perp}$$

Variation with respect to fields

Parallel component of magnetic potential

$$\frac{\delta}{\delta A_{1\|}}\mathcal{L}\circ\hat{A}_{1\|} = \frac{\delta}{\delta A_{1\|}}\int\Bigg\{-\varepsilon_\delta e\left\langle\psi_1\right\rangle - \varepsilon_\delta^2 e^2\left(\frac{1}{2mc^2}\left\langle|\mathbf{A}_1|^2\right\rangle - \frac{1}{2B(\mathbf{X}_{gy})}\partial_{\mu_{gy}}\left\langle\psi_1^2\right\rangle\right) \tag{85}$$

$$-\frac{1}{8\pi}\left|\nabla\times[\mathbf{A}_0(x)+\varepsilon_\delta\mathbf{A}_1(x,t)]\right|^2 + \varepsilon_\delta\frac{2}{c}\lambda(x,t)\nabla\cdot\mathbf{A}_1(x,t)\Bigg\}\,d\Omega$$

$$\frac{\varepsilon_\delta}{4\pi}\int dV\nabla\times\hat{A}_{1\|}\cdot B - \frac{\varepsilon_\delta^2}{4\pi}\int dV\nabla\times\hat{A}_{1\|}\cdot\nabla\times A_{1\|} +$$
$$\varepsilon_\delta e\int d\Omega F\frac{v_{\|gy}}{c}\left\langle\hat{A}_{1\|}\right\rangle - \varepsilon_\delta^2\frac{e^2}{mc^2}\int d\Omega F\left\langle A_{\|1}\hat{A}_{\|1}\right\rangle -$$
$$\varepsilon_\delta^2\int d\Omega F\frac{v_{\|gy}}{c}\frac{e^2}{B}\partial_{\mu_{gy}}\left(\left\langle\psi_1\hat{A}_{1\|}\right\rangle - \left\langle\psi_1\right\rangle\left\langle\hat{A}_{1\|}\right\rangle\right) = 0$$

Variation with respect to fields

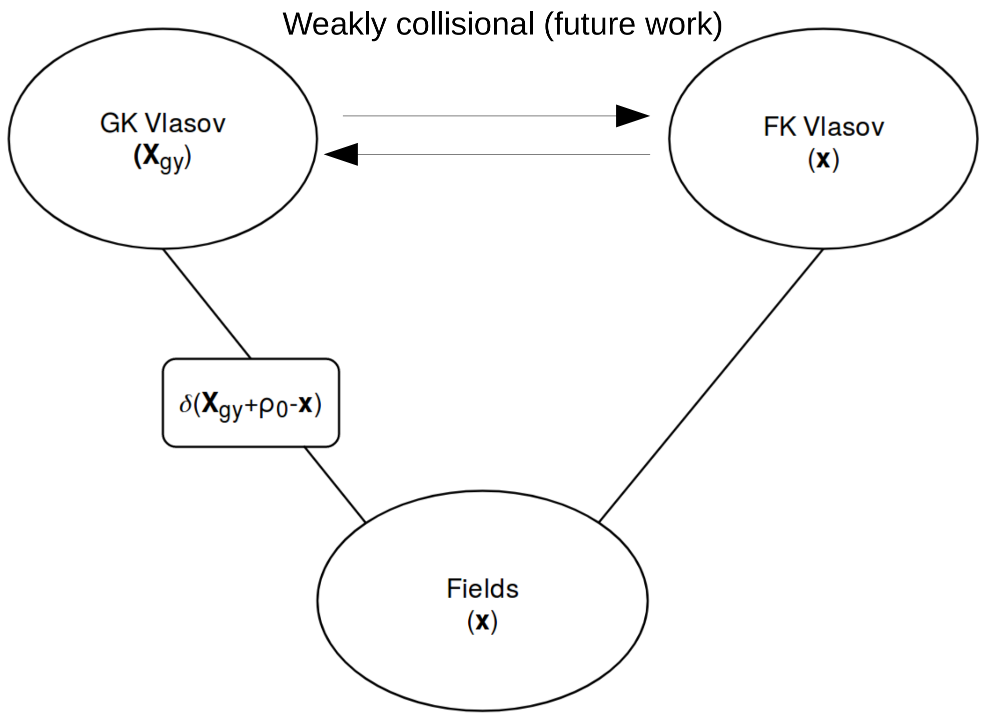
And the perpendicular component of the Ampere law is

$$\frac{\varepsilon_\delta^2}{4\pi}\int dV(\nabla\times A_1)(\nabla\times\hat{A}_1)+\varepsilon_\delta e\int d\Omega F\sqrt{\frac{2\mu_{gy}B}{mc^2}}<\hat{\perp}\cdot\hat{A}_{1\perp}>$$
$$-\frac{\varepsilon_\delta me^2}{mc^2}<A_1\cdot\hat{A}_{1\perp}>-\varepsilon_\delta^2\int d\Omega F\frac{e}{B}\partial_{\mu_{gy}}\left(\sqrt{\frac{2\mu_{gy}B}{mc^2}}\left(<\psi_1\hat{\perp}\hat{A_{1\perp}}>-<\psi_1><\hat{\perp}\cdot\hat{A}_{\perp 1}>\right)\right)+$$
$$\mathcal{O}(\varepsilon_\delta^3)=0$$

or

$$\nabla^2 A-\varepsilon_\delta\nabla^2 A_1=\varepsilon_\delta\int d\Omega eF\left\{\frac{1}{\varepsilon_\delta}\sqrt{\frac{2\mu_{gy}B}{mc^2}}<\hat{A}_{1\perp}>-\frac{e^2}{mc^2}<A_1\cdot\hat{A}_{1\perp}>-\frac{1}{B}\partial_{\mu_{gy}}\sqrt{\frac{2\mu_{gy}B}{mc^2}}<\psi_1\cdot\hat{A}_{1\perp}>\right\}$$
$$=j_{gk}(X_{gy},v_{\parallel gy},\mu_{gy},\theta)$$

Ions dynamics is connected to electron through field equations



Side note on numerical implementation
Finite Element Exterior Calculus

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl},\Omega) & \xrightarrow{\text{curl}} & H(\text{div},\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\ \downarrow \Pi_0 & & \downarrow \Pi_1 & & \downarrow \Pi_2 & & \downarrow \Pi_3 \\ V_0 & \xrightarrow{\text{grad}} & V_1 & \xrightarrow{\text{curl}} & V_2 & \xrightarrow{\text{div}} & V_3 \end{array}$$

Finite Element spaces defined as discrete de Rham complex, each of the spaces being a subset of the corresponding continuous space.

Algebraic topology and differential topology mechanism/apparatus used to describe topological information about smooth manifolds specifically suited to computation and representation of cohomology classes.

Abelian groups associated to Topological spaces + differential forms

Geometric methods for the physics of magnetised plasmas, Eric Sonnendrucker

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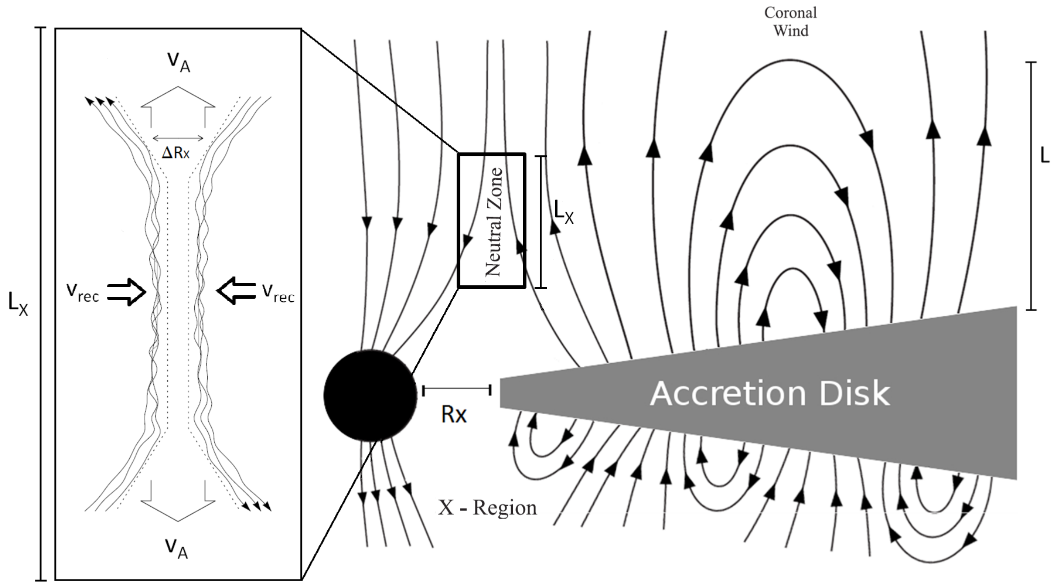
Trivially complicated, but...

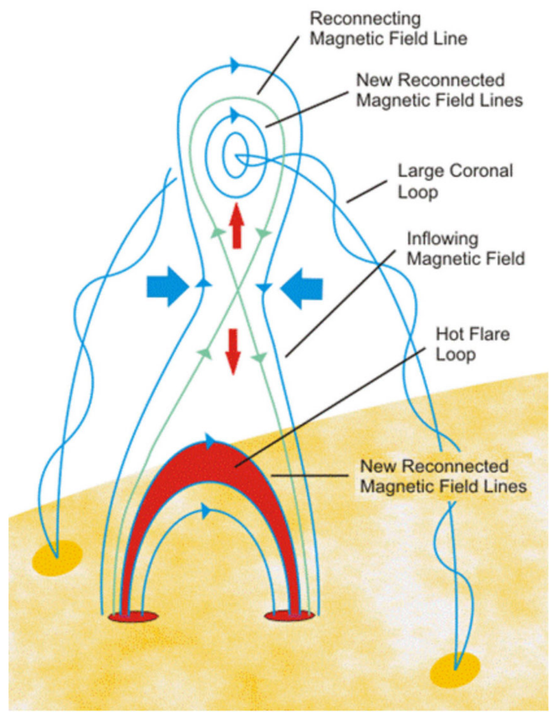
Symplectomorphic*

*Fancy word for structure preserving

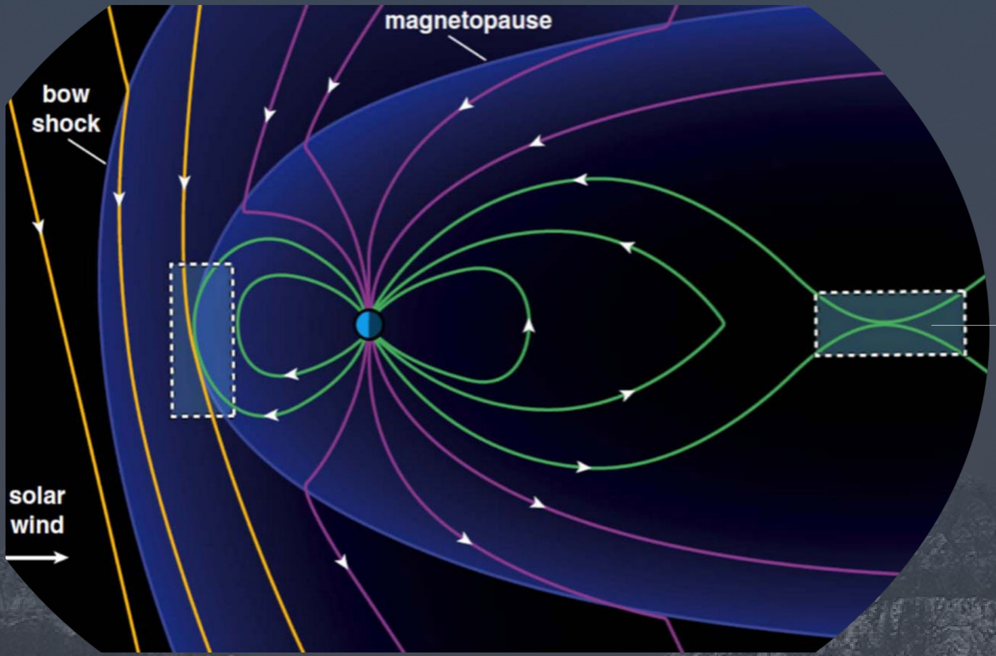
Still on the topic of gyrokinetic coordinate reduction for astrophysical plasmas...

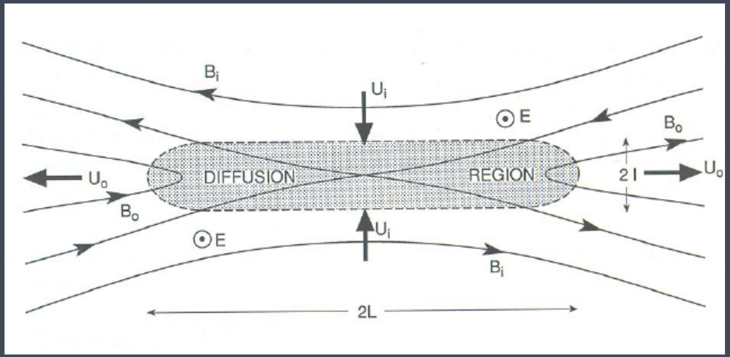


Fermi process → particle acceleration

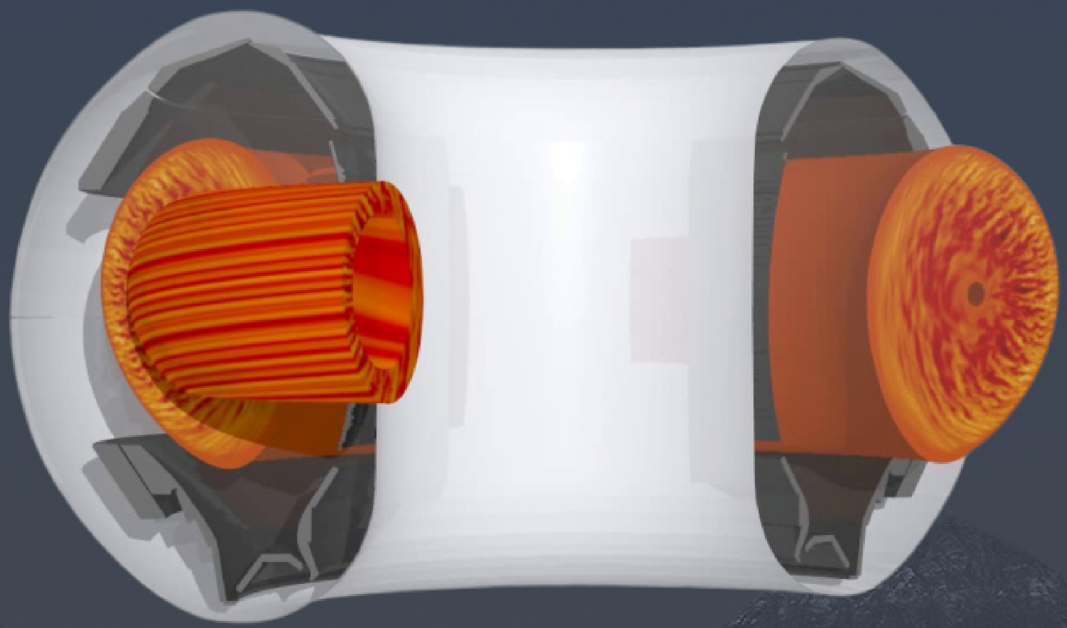


Inter-heliospheric models





The GENE code

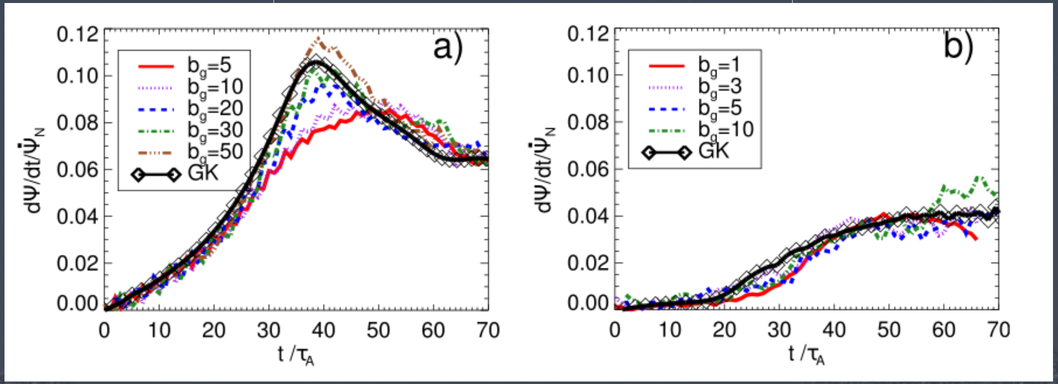


- Gyrokinetic Electromagnetic Numerical Experiment.
- Open source plasma microturbulence code.
- Compute gyroradius-scale fluctuations and the resulting transport coefficients in magnetized fusion/astrophysical plasmas.
- GENE has been used, among other things, to address both fundamental issues in plasma turbulence research and to perform comparisons with tokamak and stellarator experiments.

Precedential research...

β~0.1

β~1.0

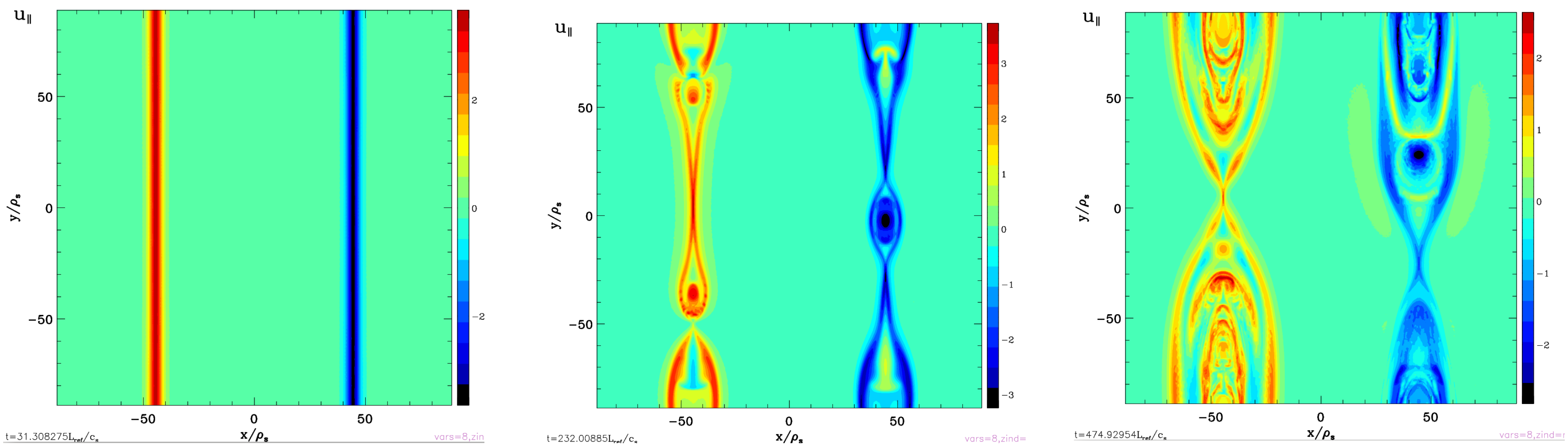


Physics of Plasmas 22, 082110 (2015)

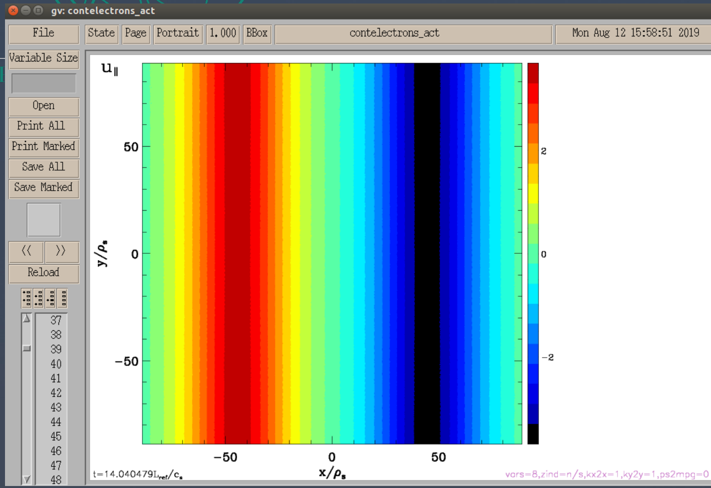
- Reconnection rates for a fully kinetic codes with different guide fields and the gyrokinetic code GENE

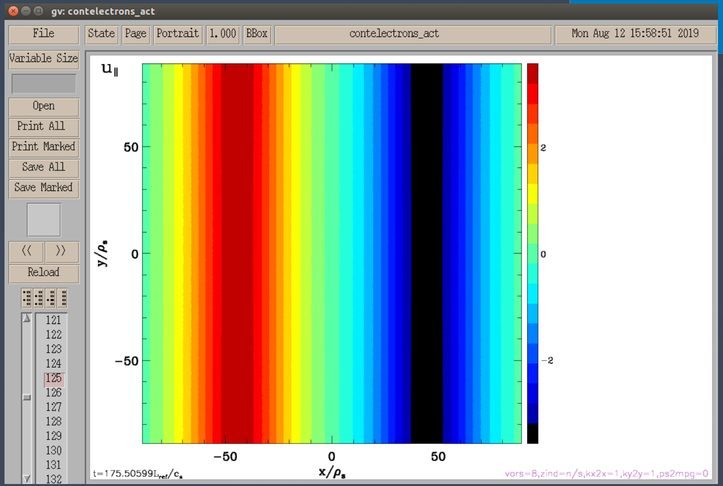
"Energy transfer and electron energization in collisionless magnetic reconnection for different guide-field intensities"
Pucci et al. Phys. Plasmas 25, 122111 (2018)

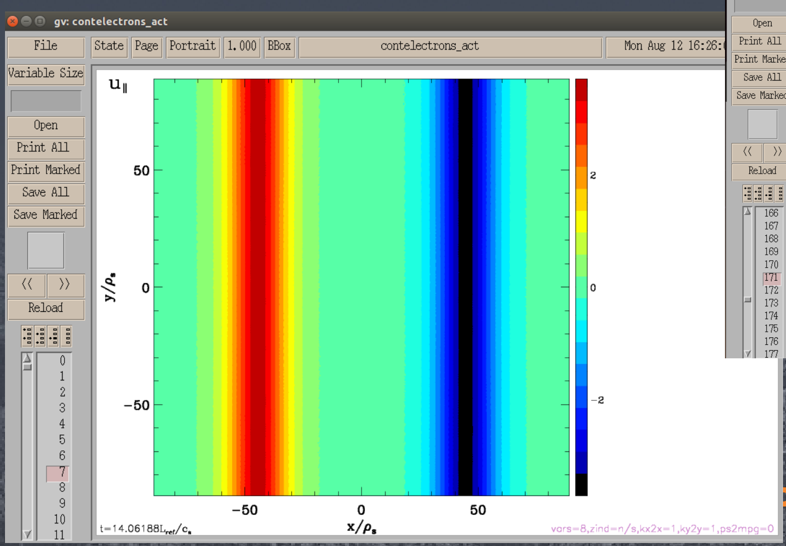
Contour plots of the electron parallel flow

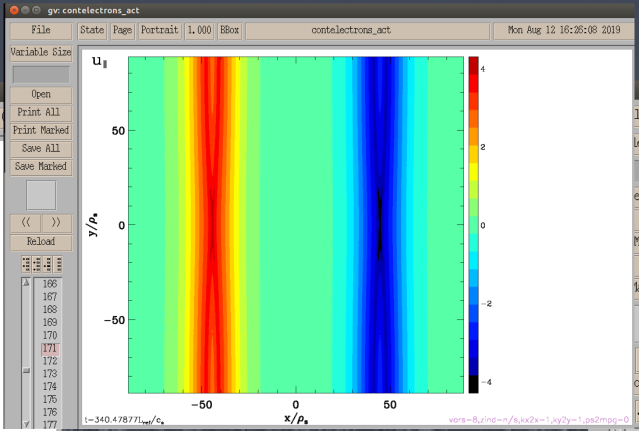


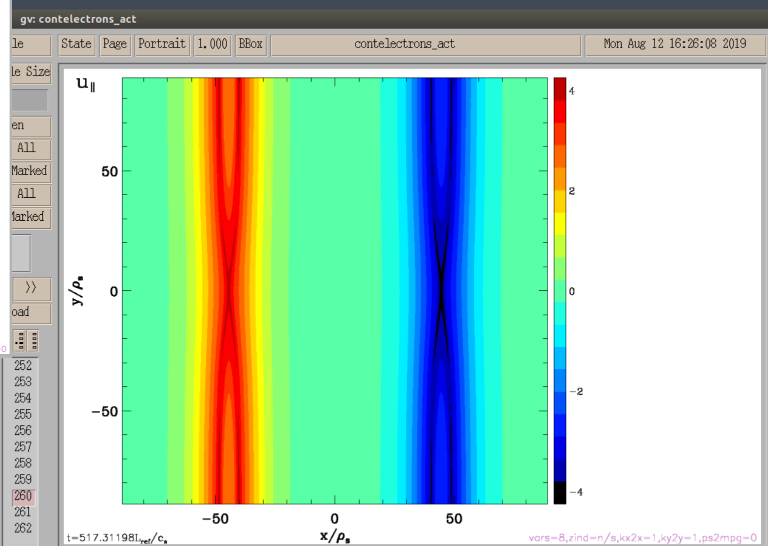
Current sheet depth analysis

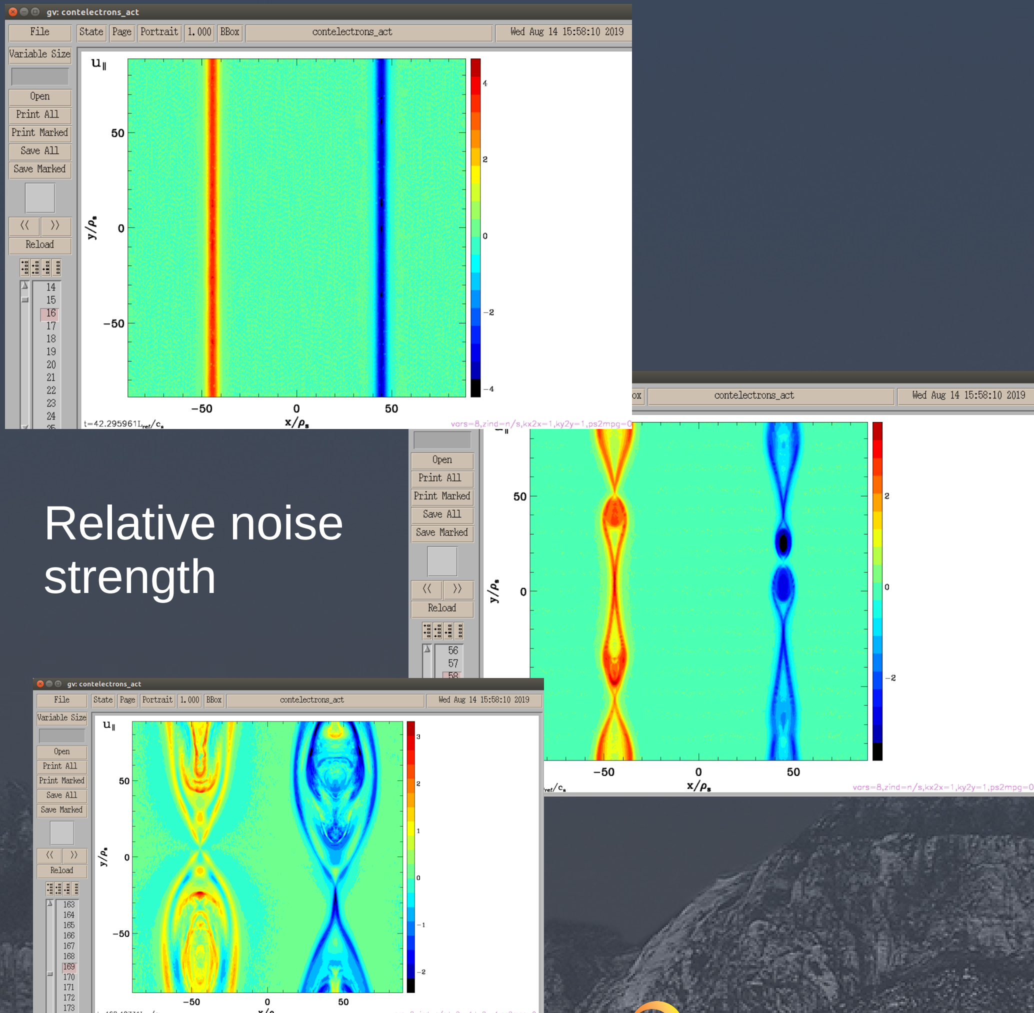




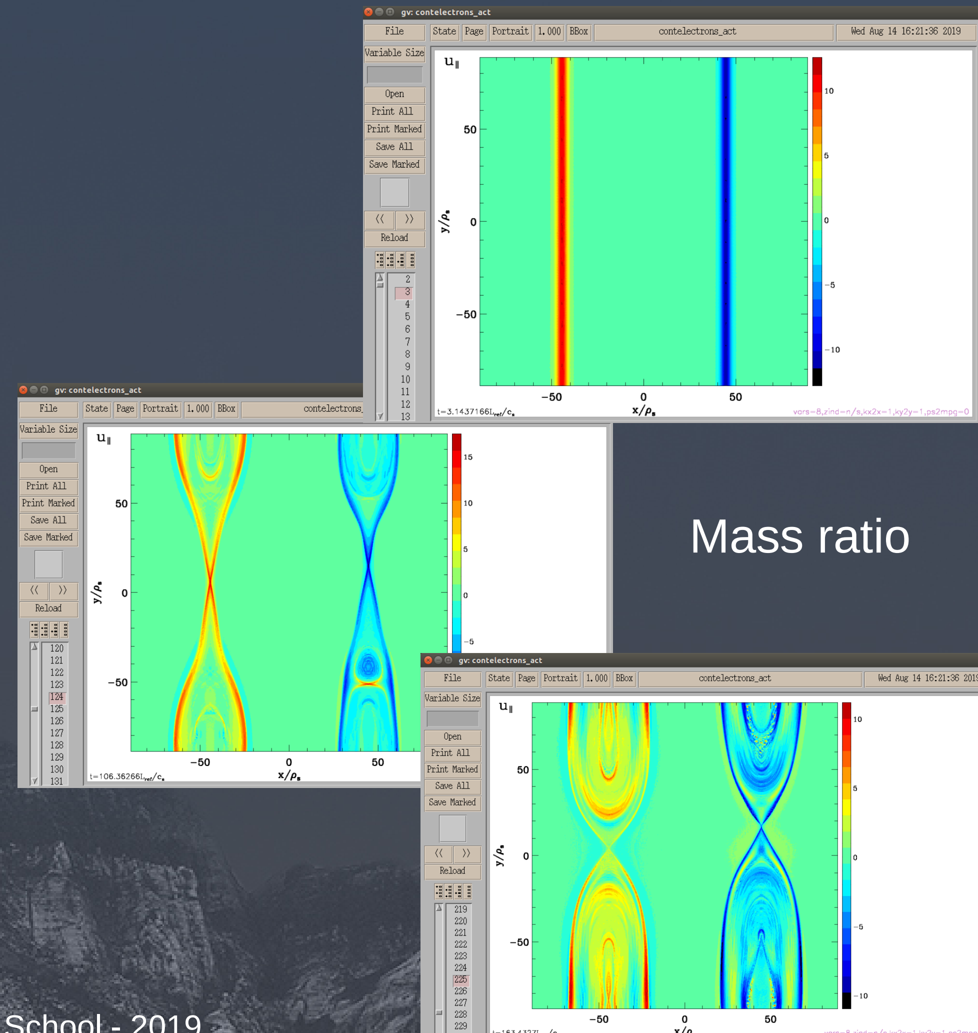








Relative noise strength



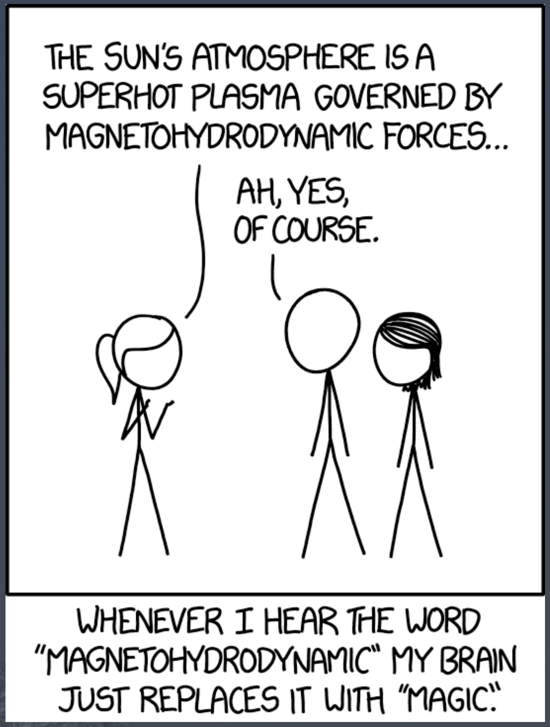
Mass ratio

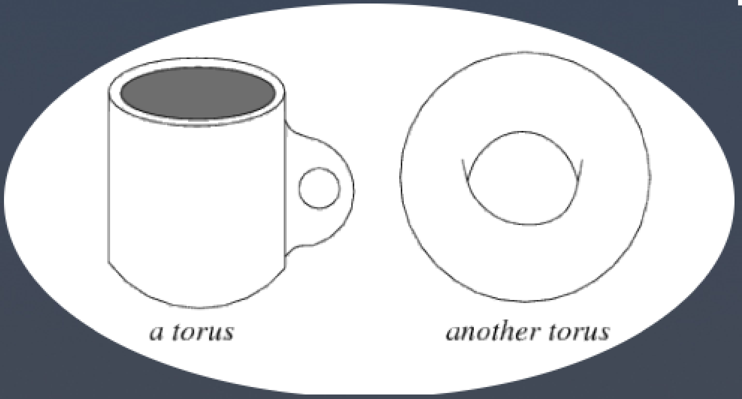
MAX-PLANCK-GESELLSCHAFT

Summary and outlook

- Consistent theoretical framework to describe kinetic physics of Space and Astrophysical plasma, as well as tokamak edge.
- Geometric derivation ensures symplectic structure preservation → Symplectomorphism.
- Gyrokinetic electrons reduce computing time whilst preserving kinetic effects.
- Fully kinetic ions interact with gyrokinetic electrons through electromagnetic field equations.
- Next steps: Discretization and numerical implementation! (+further reconnection results)

Thank you





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